# COMPACTIFICATION OF PARAHERMITIAN SYMMETRIC SPACES AND ITS APPLICATIONS, II: STRATIFICATIONS AND AUTOMORPHISM GROUPS

SOJI KANEYUKI

ABSTRACT. A simple parahermitian symmetric space is a symplectic symmetric space of a simple Lie group $G$ with two invariant Lagrangian foliations. Such a symmetric space has a nice $G$-equivariant compactification. In this paper, we obtain the stratification of the compactification, whose strata are $G$-orbits. By using this, we determine the automorphism group of the double foliation for each simple parahermitian symmetric space.

## INTRODUCTION

Let $M$ be a smooth manifold. A pair $(F^{\pm}, \omega)$ is called a *parakähler structure* (or *bi-Lagrangian structure*) on $M$ if $\omega$ is a symplectic form on $M$ and $F^{\pm}$ are two Lagrangian foliations. A significant property of parakähler structures is that a coadjoint orbit of a semisimple Lie group is hyperbolic if and only if it admits an invariant parakähler structure ([4]). A symmetric space $G/H$ of a Lie group $G$ is called a *parahermitian symmetric space* (or *bi-Lagrangian symmetric space*)([10]) if $G/H$ admits a $G$-invariant parakähler structure $(F^{\pm}, \omega)$. The simplest example of parahermitian symmetric spaces is the symmetric space $\mathrm{SL}(2,\mathbb{R})/\mathbb{R}^*$, realized as the one-sheeted hyperboloid $x^2+y^2-z^2=1$ in $\mathbb{R}^3 = \mathrm{Lie}\,\mathrm{SL}(2,\mathbb{R})$. The Lagrangian foliations $F^{\pm}$ are given by the two families of rulings of the hyperboloid. Parahermitian symmetric spaces of semisimple Lie groups were classified and characterized group-theoretically in [10, 5]. A semisimple symmetric space $G/H$ is parahermitian if and only if $H$ is an open subgroup of the Levi subgroup of a parabolic subgroup with abelian nilradical. Semisimple parahermitian symmetric spaces $G/H$ are in one-to-one correspondence (up to covering) with semisimple graded Lie algebras (shortly, GLAs) of the 1st kind $\mathfrak{g} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$, in such a way that $\mathfrak{g} = \mathrm{Lie}\,G$ and $\mathfrak{g}_0 = \mathrm{Lie}\,H$. For the explicit forms of simple parahermitian symmetric pairs, see the tables in 6.4 and 7.2.

Now let $\mathfrak{g} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$ be a simple GLA, and let $M = G/G_0$ be the parahermitian symmetric space corresponding to the symmetric pair $(\mathfrak{g}, \mathfrak{g}_0)$, where $G$ is the largest possible open subgroup of the automorphism group of $\mathfrak{g}$ such that $G/G_0$ is realized as an $\mathrm{Ad}\,\mathfrak{g}$-orbit in $\mathfrak{g}$. The subgroup $U^{\pm} := G_0 \exp \mathfrak{g}_{\pm 1}$ are the parabolic subgroups with $\mathrm{Lie}\,U^{\pm} = \mathfrak{g}_0 + \mathfrak{g}_{\pm 1}$. The flag manifolds $M^{\pm} = G/U^{\pm}$ are symmetric $R$-spaces. Let $r$ be the ranks of $M^{\pm}$. Then there are exactly $r$ numbers of $\mathfrak{sl}(2,\mathbb{R})$-triplets in $\mathfrak{g}$ which are pairwise commutative and whose direct sum is expressed as a graded subalgebra $\mathfrak{a}_{-1} + \mathfrak{a}_0 + \mathfrak{a}_1$ in $\mathfrak{g}$ (cf. [8]). One has the root system $\Delta(\mathfrak{g}, \mathfrak{a}_0)$ of $\mathfrak{g}$ with respect to the abelian subspace $\mathfrak{a}_0$. $\Delta(\mathfrak{g}, \mathfrak{a}_0)$ is of $BC_r$-type or $C_r$-type ([8, 1]). We say that $G/G_0$ and the GLA $\mathfrak{g}$ are of $BC_r$-type or $C_r$-type, if $\Delta(\mathfrak{g}, \mathfrak{a}_0)$ is.

Research at MSRI is supported in part by NSF grant DMS-9810361.

A fundamental problem of the geometry of parahermitian symmetric spaces ($M = G/H$, $F^{\pm}$, $\omega$) is to determine the automorphism group $\mathrm{Aut}(M, F^{\pm})$—the group consisting of diffeomorphisms of $M$ leaving the double foliation $F^{\pm}$ invariant. The aim of this paper is to settle this problem for an arbitrary simple Lie group $G$. A partial answer was given by Tanaka [15] under the assumption that $G$ is classical simple. Let us describe our procedure to determine the automorphism group. The first step is to obtain the $G$-orbit structure of $\tilde{M} = M^- \times M^+$, which is the natural $G$-equivariant compactification of $M$ (cf. [6] and Sections 2 and 3). The second step is to show that the $G$-orbit decomposition gives $\tilde{M}$ a stratification whose strata are $G$-orbits. This is done in Sections 5 and 6.

The third step is concerned with $BC_r$-type. We now assume that $M$ is of $BC_r$-type. In terms of the root system $\Delta(\mathfrak{g}, \mathfrak{a}_0)$, we construct a grading $\mathfrak{g} = \sum_{k=-2}^{2} \mathfrak{g}_k(r)$ of the 2nd kind having the property that $\mathfrak{g}_{\pm 1}(r)$ are expressed as the direct sum of two equi-dimensional abelian subspaces, $\mathfrak{g}^+_{\pm 1}(r) + \mathfrak{g}^-_{\pm 1}(r)$ (cf. (4.10),(4.14)). Such a grading is called a *pseudo-product grading* of $\mathfrak{g}$ (Tanaka [15]). Let $Q_r$ be the isotropy subgroup of $G$ at the base point of the minimal dimensional $G$-orbit $M_0$. The third step is to show that $Q_r$ is the parabolic subgroup with $\mathrm{Lie}\, Q_r = \mathfrak{g}_{-2}(r) + \mathfrak{g}_{-1}(r) + \mathfrak{g}_0(r)$ (Proposition 4.6) and that its Levi subgroup coincides with the automorphism group of the pseudo-product grading (cf. Lemma 7.4 and Remark 7.1). The flag manifold $M_0 = G/Q_r$ has the double foliation $F_0^{\pm}$ induced from the product structure of $\tilde{M}$. We denote by $\mathrm{Aut}(M_0, F_0^{\pm})$ the group of diffeomorphisms of $M_0$ leaving $F_0^{\pm}$ invariant. Then the Tanaka's theory [15] of Cartan connections for pseudo-product manifolds, together with the third step assertion guarantees the validity of the relation $\mathrm{Aut}(M_0, F_0^{\pm}) = G$.

For the case where $M$ is of $C_r$-type, the minimal $G$-orbit $M_0$ coincides with $G/U^- = M^-$, and the double foliation $F_0^{\pm}$ becomes trivial. But, in turn, $M^-$ has the generalized conformal structure $\mathcal{K}$, which is obtained from the cone defined as the union of singular $G_0$-orbits in $\mathfrak{g}_1$ ([2]). We determined in [2] the conformal automorphism group $\mathrm{Aut}(M^-, \mathcal{K})$ for each symmetric $R$-space $M^-$.

The fourth or the last step is to obtain the injective homomorphism of $\mathrm{Aut}(M, F^{\pm})$ to $\mathrm{Aut}(M_0, F_0^{\pm})$ or to $\mathrm{Aut}(M^-, \mathcal{K})$. For this purpose, the stratification of $\tilde{M}$ is essential. Let $f \in \mathrm{Aut}(M, F^{\pm})$. Then $f$ extends to $\tilde{M}$ as an automorphism $\tilde{f}$ of the product structure of $\tilde{M}$. It follows that $\tilde{f}$ is an automorphism of the stratification of $\tilde{M}$ (Corollary 6.14). In particular $\tilde{f}$ leaves $M_0$ stable. It is shown that the assignment $f \mapsto \tilde{f}|_{M_0}$ gives the injective homomorphism as desired. The main results are Theorems 8.1 and 8.4.

We want to supplement some details on the stratification of $\tilde{M}$, since it is a rather independent topic. By a stratification of a real analytic manifold $X$, we mean a partition $X = \coprod_{k=0}^{s} A_k$ which satisfies the following conditions: (S1) Each $A_k$ is an analytic submanifold of $X$, (S2) the closure $\overline{A_k}$ of $A_k$ is an analytic set of $X$ and coincides with $A_{\leq k} := \coprod_{i=0}^{k} A_i$ $(0 \leq k \leq s)$, and (S3) the singular locus $\mathrm{Sing}(\overline{A_k})$ is given by $A_{\leq k-1}$ $(1 \leq k \leq s-1)$. Let $\tilde{M} = \coprod_{k=0}^{r} M_k$ be the $G$-orbit decomposition ($G$ acts on $\tilde{M}$ diagonally), where $\dim M_k > \dim M_{k-1}$ and $M_r = M$ is open dense in $\tilde{M}$. For the $G$-orbit decomposition of $\tilde{M}$, the properties (S1) and (S2) were already proved in [6]. We will verify (S3) in this paper.

Suppose first that the $GLA$ $\mathfrak{g}$ is of $BC_r$-type. We consider the two abelian subspaces of $\mathfrak{g}$: $\mathfrak{g}_1 = \mathfrak{g}_2(r) + \mathfrak{g}_1^-(r)$, $\mathfrak{g}_1' := \mathfrak{g}_2(r) + \mathfrak{g}_1^+(r)$. The direct sum $\mathfrak{g}_1 \oplus \mathfrak{g}_1'$ is imbedded in $\tilde{M}$ as an open subset. We identify $\mathfrak{g}_1 \oplus \mathfrak{g}_1'$ with its image in $\tilde{M}$. Let $M_k^* := M_k \cap (\mathfrak{g}_1 \oplus \mathfrak{g}_1')$, which is open

dense in $M_k$. The closure $\overline{M_k^*}$ of $M_k^*$ in $\mathfrak{g}_1 \oplus \mathfrak{g}_1'$ coincides with $M_{\leq k}^* := \coprod_{i=0}^k M_i^*$ and it is an algebraic variety in $\mathfrak{g}_1 \oplus \mathfrak{g}_1'$ (Theorem 5.13). Obviously we have $\mathrm{Sing}(M_{\leq k}) = G(\mathrm{Sing}(M_{\leq k}^*))$. Thus, in order to find the singular locus of $M_{\leq k}$, it is enough to find that of $M_{\leq k}^*$. Now we look at $\mathfrak{g}_2(r)$. The Levi subgroup $L$ of $Q_r$ corresponding to $\mathfrak{g}_0(r)$ acts on $\mathfrak{g}_2(r)$. We have a partition $\mathfrak{g}_2(r) = \coprod_{k=0}^r V_k$, where $V_k$ is a union of equi-dimensional $L$-orbits with $\dim V_k > \dim V_{k-1}$. As a conclusion, the above partition is a stratification of $\mathfrak{g}_2(r)$. The validity of (S1) and (S2) was proved in [8]; as was shown in [8], $V_{\leq k}$ $(0 \leq k \leq r-1)$ is an algebraic variety in $\mathfrak{g}_2(r)$. By Proposition 6.3, the problem of finding the singular locus of $M_{\leq k}^*$ is reduced to finding that of $V_{\leq k}$. In the case where the *GLA* $\mathfrak{g}$ is of $C_r$-type, we have $\mathfrak{g}_1 = \mathfrak{g}_1' = \mathfrak{g}_2(r)$. Therefore it is enough to concern the determinantal varieties $V_{\leq k}$ in $\mathfrak{g}_1$ for the case where the GLA $\mathfrak{g}$ is of $C_r$-type. In the realization of $\mathfrak{g}_1$ as a matrix space, $V_{\leq k}$ is a complex or real determinantal variety of classical or exceptional type. The determination of $\mathrm{Sing}(V_{\leq k})$ will be carried out in Section 6. Levasseur-Stafford [11] is a good reference for the complex classical case. The final result on the stratification of $\tilde{M}$ is given by Theorem 6.13.

The class of simple parahermitian symmetric spaces of $C_r$-type contains an interesting sub-class of symmetric spaces of Cayley type, which are causal symmetric spaces. For a symmetric space $M$ of Cayley type, $\tilde{M}$ is the causal compactification of $M$ (cf. [12]). As an application of the results of the present paper, one can determine the full causal automorphism group of $M$. In the forth-coming paper, we will treat this topic.

The author would like to thank MSRI, Berkeley, where most of this work was completed during his stay in the fall, 2001. The author happily express his thanks to Simon Gindikin for frequent valuable conversations.

The paper is organized as follows:

1. Preliminaries on parahermitian symmetric spaces.
2. Double foliation of $M$.
3. Orbit structure of $\tilde{M}$.
4. Isotropy subgroups for boundary orbits.
5. Siegel-type realization of orbits.
6. Stratification of $\tilde{M}$.
7. Double foliation on the minimal boundary orbit.
8. Determination of automorphism groups of $M$.

## 1. Preliminaries on parahermitian symmetric spaces

1.1. Let $M$ be a connected $2n$-dimensional smooth manifold, and let $F^{\pm}$ be two $n$-dimensional completely integrable distributions on $M$. $(F^{\pm})$ is called a *paracomplex structure* on $M$ ([10]) if the tangent bundle $TM$ of $M$ can be expressed as the Whitney sum $F^+ \oplus F^-$. In this case $(M, F^{\pm})$ is called a *paracomplex manifold*. A paracomplex manifold $(M, F^{\pm})$ is called a *parakähler manifold* ([10]) if there exists a symplectic form $\omega$ on $M$ with respect to which $F^{\pm}$ are Lagrangian subbundles. For a parakähler manifold $(M, F^{\pm}, \omega)$, one can consider the two kinds of automorphisms: By a *paracomplex automorphism* of $M$ we mean a diffeomorphism of $M$ which leaves $F^{\pm}$ invariant. By a *paracomplex isometry* of $M$ we mean a paracomplex automorphism leaving $\omega$ invariant. We denote by $\mathrm{Aut}(M, F^{\pm})$ (resp. $\mathrm{Aut}(M, F^{\pm}, \omega)$) the group of paracomplex automorphisms (resp. paracomplex isometries) of $M$. The group $\mathrm{Aut}(M, F^{\pm}, \omega)$ is always a finite-dimensional Lie group, but $\mathrm{Aut}(M, F^{\pm})$ is not in general.

**Definition 1.1** ([10])**.** Let $M = G/H$ be an almost effective symmetric coset space of a Lie group $G$, and let $(F^{\pm}, \omega)$ be a parakähler structure on $M$. If $G$ acts on $M$ as paracomplex isometries with respect to $(F^{\pm}, \omega)$, then $(M = G/H,\ F^{\pm},\ \omega)$ is called a *parahermitian symmetric space*.

For each parahermitian symmetric space $M = G/H$, the Lie algebra $\mathfrak{g} = \operatorname{Lie} G$ has the structure of a GLA of the first kind $\mathfrak{g} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$ ([10]). Under the assumption of semisimplicity of $\mathfrak{g}$, the assignment $M \rightsquigarrow \mathfrak{g} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$ induces a bijection between the set of local isomorphism classes of parahermitian symmetric spaces and the set of isomorphism classes of effective semisimple GLA of the first kind ([5]). In this case the original parakähler structure on $M$ can be recovered by the grading of $\mathfrak{g}$.

1.2. Let us start with a real simple GLA of the first kind

$$\mathfrak{g} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1. \tag{1.1}$$

The automorphism group of the Lie algebra $\mathfrak{g}$ is denoted by $\operatorname{Aut}\mathfrak{g}$. Let $(Z, \tau)$ be the associated pair of the GLA: $Z$ is the *characteristic element* of the GLA, that is, $Z$ is a unique element of $\mathfrak{g}_0$ satisfying the condition $\operatorname{ad} Z = k1$ on $\mathfrak{g}_k$, $k = 0, \pm 1$, and $\tau$ is a Cartan involution of $\mathfrak{g}$ satisfying $\tau(Z) = -Z$. Let $\sigma = \operatorname{Ad}\exp \pi i Z$.Then $\sigma$ is the involutive automorphism of $\mathfrak{g}$ such that $\sigma = 1$ on $\mathfrak{g}_0$ and $-1$ on $\mathfrak{m} := \mathfrak{g}_{-1} + \mathfrak{g}_1$. Thus we have a symmetric triple $(\mathfrak{g}, \mathfrak{g}_0, \sigma)$. Let $G_0$ be the centralizer of $Z$ in $\operatorname{Aut}\mathfrak{g}$, and let $G$ be the open subgroup of $\operatorname{Aut}\mathfrak{g}$ generated by $G_0$ and $\operatorname{Ad}\mathfrak{g}$. Note that $\operatorname{Lie} G_0 = \mathfrak{g}_0$ and that $G_0$ coincides the group of grade-preserving automorphisms of the GLA (1.1).

**Proposition 1.2.** *The coset space $M = G/G_0$ is a parahermitian symmetric space corresponding to the symmetric triple $(\mathfrak{g}, \mathfrak{g}_0, \sigma)$. The group $G$ acts on $M$ effectively as paracomplex isometries.*

*Proof.* If we put $\tilde{\sigma}(a) = \sigma a \sigma$, $a \in G$, then $\tilde{\sigma}$ is an involutive automorphism of $\operatorname{Aut}\mathfrak{g}$. $\tilde{\sigma}$ leaves $G$ stable. Let $G_{\tilde{\sigma}}$ be the subgroup of $G$ consisting of all $\tilde{\sigma}$-fixed elements of $G$. Then, from the definition of $\sigma$, it follows that $G_0$ is an open subgroup of $G_{\tilde{\sigma}}$. So $M = G/G_0$ is a symmetric space. By the definition of $G_0$, $M$ is realized in $\mathfrak{g}$ as the adjoint $G$-orbit through $Z \in \mathfrak{g}$. Therefore the Kirillov-Kostant form

$$\tilde{\omega}(X, Y) = (Z, [X, Y]), \qquad X, Y \in \mathfrak{g} \tag{1.2}$$

induces a $G$-invariant symplectic form $\omega$ on $M$,where $(\ ,\ )$ denotes the Killing form of $\mathfrak{g}$. Let $0 \in M$ be the origin of $M = G/G_0$. We identify $\mathfrak{m}$ with the tangent space $T_0 M$ of $M$ at 0. Then the two $G_0$-invariant subspaces $\mathfrak{g}_{\pm 1}$ extend to $G$-invariant distributions $F^{\pm}$ on $M$, which are Lagrangian with respect to $\omega$ by (1.2). The complete integrability of $F^{\pm}$ has been proved in two ways, one in [10] by differential geometric argument, the other by an algebraic method using dipolarizations. To prove effectivity of the $G$-action on $M$, first note that the natural $G_0$-action on $\mathfrak{m}$ can be identified with the linear isotropy group at $0 \in M = G/G_0$. Let $a \in G$ and suppose that $a$ acts on $M$ as the identity. Then $a \in G_0$ and $a$ acts on $\mathfrak{m}$ as the identity. Since $\mathfrak{g}$ is simple, we have $\mathfrak{g}_0 = [\mathfrak{g}_1, \mathfrak{g}_{-1}]$. So it follows that $a$ acts on $\mathfrak{g}_0$ as the identity. This implies that $a$ is the unit element of $G$. □

The parahermitian symmetric space $(M = G/G_0, F^{\pm}, \omega)$ thus constructed is called the *parahermitian symmetric space associated to a simple GLA* (1.1). The parahermitian symmetric space $M = G/G_0$, which is a hyperbolic $\operatorname{Ad} G$-orbit, is the bottom space among the parahermitian symmetric spaces corresponding to the symmetric triple $(\mathfrak{g}, \mathfrak{g}_0, \sigma)$.

## 2. Double foliations of $M$

Let $(M = G/G_0, F^{\pm}, \omega)$ be the parahermitian symmetric space associated to a simple GLA (1.1). We denote by $F^{\pm}(p)$ the leaves of $F^{\pm}$ through a point $p \in M$. It is easy to see that the leaves $F^{\pm}(0)$ through the origin $0 \in M$ are given by the orbits $(\exp \mathfrak{g}_{\pm 1})\cdot 0$. Consider the parabolic subgroups $U^{\pm} := G_0 \exp \mathfrak{g}_{\pm 1}$ of $G$. Note that $\mathfrak{u}^{\pm} := \operatorname{Lie} U^{\pm} = \mathfrak{g}_0 + \mathfrak{g}_{\pm 1}$. The flag manifolds $G/U^{\pm}$ are called the *symmetric $R$-spaces associated with $M$* or with the GLA (1.1).

**Lemma 2.1.** *Let $M^{\pm}$ be the sets of leaves of $F^{\pm}$ on $M$. Then $M^{\pm}$ are identified with the flag manifolds $G/U^{\pm}$.*

*Proof.* Since $F^{\pm}$ are $G$-invariant, any element of $G$ induces a permutation on the sets of leaves of $F^{\pm}$, which means that $G$ acts on $M^{\pm}$. The transitivity of $G$ on $M^{\pm}$ follows from that of $G$ on $M$. Now let $g \in G$ and suppose for example $gF^{+}(0) = F^{+}(0)$. Then the point $g\cdot 0$ is in $F^{+}(0)$. One can write $g\cdot 0 = \exp X \cdot 0$ for some $X \in \mathfrak{g}_1$. Therefore $g^{-1}\exp X \in G_0$, which implies $g \in U^{+}$, and $M^{+}$ is expressed as $G/U^{+}$. □

**Lemma 2.2.** *For any point $p \in M$, we have $F^{+}(p) \cap F^{-}(p) = \{p\}$.*

*Proof.* One can assume $p$ to be the origin 0. Any point $q \in F^{+}(0) \cap F^{-}(0)$ can be expressed as $q = \exp X \cdot 0 = \exp Y \cdot 0$ for $X \in \mathfrak{g}_1$ and $Y \in \mathfrak{g}_{-1}$. This implies that $\exp X \in (\exp Y)G_0 \subset U^{-}$. Consequently $\exp X \in U^{+} \cap U^{-} = G_0$ and hence $\exp X \in (\exp \mathfrak{g}_1) \cap G_0 = (1)$, which implies $X = 0$. Thus we have $q = p$. □

We denote the points $F^{\pm}(0) \in M^{\pm}$ by $0^{\pm}$, and let us consider the product manifolds

$$\tilde{M} = M^{-} \times M^{+} \tag{2.1}$$

with the origin $(0^{-}, 0^{+})$. $\tilde{M}$ has a double foliation arising from the product structure. The leaves through the point $(g_1 0^{-}, g_2 0^{+})$, $g_1,\ g_2 \in G$, are denoted by $M^{\pm}(g_1 0^{-}, g_2 0^{+})$. $M^{-}(g_1 0^{-}, g_2 0^{+})$ is called the horizontal leaf and $M^{+}(g_1 0^{-}, g_2 0^{+})$ is called the vertical leaf. They are given by

$$\begin{aligned} M^{-}(g_1 0^{-}, g_2 0^{+}) &= G g_1 0^{-} \times \{g_2 0^{+}\} = G/g_1 U^{-} g_1^{-1} \times \{g_2 0^{+}\}, \\ M^{+}(g_1 0^{-}, g_2 0^{+}) &= \{g_1 0^{-}\} \times G g_2 0^{+} = \{g_1 0^{-}\} \times G/g_2 U^{+} g_2^{-1}. \end{aligned} \tag{2.2}$$

Let us define a map $\varphi$ of $M$ to $\tilde{M}$ by putting

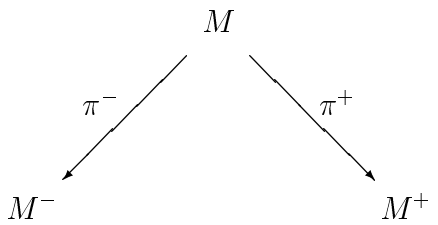
$$\varphi(p) = \big(F^{-}(p), F^{+}(p)\big), \qquad p \in M. \tag{2.3}$$

**Lemma 2.3.** *$\varphi$ is a $G$-equivariant open imbedding of $M$ into $\tilde{M}$ and preserves the double foliations on $M$ and $\tilde{M}$; actually we have $\varphi\big(F^{\mp}(p)\big) \subset M^{\pm}\big(\varphi(p)\big)$, $p \in M$.*

*Proof.* Let $g \in G$. Then $\varphi(g\cdot 0) = \big(F^{-}(g\cdot 0), F^{+}(g\cdot 0)\big) = (g\cdot 0^{-}, g\cdot 0^{+})$. From this and Lemma 2.2 it follows that $\varphi$ is $G$-equivariant imbedding. The openness of $\varphi$ follows from dimension counting. Now let $q \in F^{-}(p)$. Then $F^{-}(q) = F^{-}(p)$ and hence $\varphi(q) = \big(F^{-}(p), F^{+}(q)\big)$, which implies that $\varphi(q)$ lies on the vertical leaf through $\varphi(p)$. □

Since $U^{-} \cap U^{+} = G_0$, $M$ has the structure of the double fibration over $M^{\pm}$. The projections $\pi^{\pm} \colon M \to M^{\pm}$ are given by

$$\pi^{\pm}(g \cdot 0) = g \cdot 0^{\pm}, \qquad g \in G. \tag{2.4}$$




**Lemma 2.4.** *For each point $p \in M$, we have*

$$F^{\pm}(p) = (\pi^{\pm})^{-1}\big(\pi^{\pm}(p)\big).$$

*Proof.* It is enough to prove the assertion for the case where $p$ is the origin. Choose a point $(\exp X)0 \in F^+(0),\ \ X \in \mathfrak{g}_1$. Then we have $\pi^+\big((\exp X)0\big) = (\exp X)0^+ = 0^+$, and hence $(\exp X)0 \in (\pi^+)^{-1}(0^+) = (\pi^+)^{-1}\big(\pi^+(0)\big)$. Conversely, let $p \in (\pi^+)^{-1}(0^+)$. Then $\pi^+(p) = 0^+$. Consequently $p$ can be written as $p = u0$, where $u \in U^+$. If we write $u = (\exp Y)h,\ \ Y \in \mathfrak{g}_1,\ h \in G_0$, then we have $p = (\exp Y)0 \in F^+(0)$. □

If we denote the projections by $\varpi^{\pm}\colon \tilde{M} \to M^{\pm}$, then we have $\varpi^{\pm} \cdot \varphi = \pi^{\pm}$ (cf. (2.4)). Therefore, under the identification of $M$ with $\varphi(M)$, the double fibration of $M$ is the restriction of the trivial double fibration of $\tilde{M}$ to $M$. Later on we always identify $M$ with its $\varphi$-image in $\tilde{M}$. As was seen in the proof of Lemma 2.3, $M$ is an orbit through the origin $(0^-, 0^+) \in \tilde{M}$ under the diagonal $G$-action.

## 3. Orbit structure of $\tilde{M}$

3.1. We wish to consider the orbit structure of $\tilde{M}$ under the diagonal $G$-action. We start with simple GLA (1.1). Recall the decomposition $\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{m}$ by $\sigma$ (cf. §1). Also we have the Cartan involution $\tau$ satisfying $\tau(Z) = -Z$. The property $\tau(Z) = -Z$ means that $\tau$ is grade-reversing, i.e., $\tau(\mathfrak{g}_k) = \mathfrak{g}_{-k},\ \ k = 0, \pm 1$. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition by $\tau$, where $\tau = 1$ on $\mathfrak{k}$ and $-1$ on $\mathfrak{p}$. Since $\sigma$ and $\tau$ commute, we have the decomposition

$$\mathfrak{g} = \mathfrak{k}_0 + \mathfrak{m}_{\mathfrak{k}} + \mathfrak{p}_0 + \mathfrak{m}_{\mathfrak{p}}, \tag{3.1}$$

where $\mathfrak{k}_0 = \mathfrak{g}_0 \cap \mathfrak{k},\ \ \mathfrak{m}_{\mathfrak{k}} = \mathfrak{m} \cap \mathfrak{k},\ \ \mathfrak{p}_0 = \mathfrak{g}_0 \cap \mathfrak{p}$ and $\mathfrak{m}_{\mathfrak{p}} = \mathfrak{m} \cap \mathfrak{p}$. Note that $Z \in \mathfrak{p}_0$. Choose a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ such that $Z \in \mathfrak{a}$. Then $\mathfrak{a}$ is contained in $\mathfrak{p}_0$. Let $\Delta$ be the root system of $\mathfrak{g}$ with respect to $\mathfrak{a}$. Let $(\ \ ,\ \ )$ denote the Killing form of $\mathfrak{g}$. Then we have the partition of $\Delta$ corresponding to the grading of $\mathfrak{g}$:

$$\begin{aligned} &\Delta = \Delta_{-1} \amalg \Delta_0 \amalg \Delta_1, \\ &\Delta_k = \{\, \alpha \in \Delta : (\alpha, Z) = k \,\}, \qquad k = 0, \pm 1. \end{aligned} \tag{3.2}$$

Choose a linear order in $\Delta$ in such a way that $\Delta_1 \subset \Delta^+ \subset \Delta_0 \cup \Delta_1$, where $\Delta^+$ denotes the positive system of $\Delta$ with respect to that order. Then choose a maximal system of strongly orthogonal roots, $\Gamma = \{\beta_1, \ldots, \beta_r\}$ in $\Delta_1$, such that each $\beta_i$ has the same length and that $\theta = \beta_1 > \beta_2 > \cdots > \beta_r,\ \ \theta$ being the highest root in $\Delta$. Here the number $r$ is the split rank of the symmetric pair $(\mathfrak{g}, \mathfrak{g}_0)$. Note that $r$ is equal to the rank of the symmetric $R$-space $G/U^-$.

Moreover choose a root vector $E_i$ in the root space $\mathfrak{g}^{\beta_i} \subset \mathfrak{g}_1\ \ (1 \le i \le r)$ in such a way that

$$[E_i, E_{-i}] = \check{\beta}_i = \frac{2}{(\beta_i, \beta_i)}\beta_i, \qquad 1 \le i \le r,$$

where $E_{-i} = -\tau(E_i) \in \mathfrak{g}^{-\beta_i} \subset \mathfrak{g}_{-1}$. Put $X_i := E_i + E_{-i} \in \mathfrak{m}_\mathfrak{p}$ and $Y_i := E_i - E_{-i} \in \mathfrak{m}_\mathfrak{k}$ $(1 \leq i \leq r)$. Then $\mathfrak{c} = \sum_{i=1}^r \mathbb{R}X_i$ is a maximal abelian subspace of $\mathfrak{m}_\mathfrak{p}$. $\mathfrak{c}$ is a split Cartan subalgebra of the symmetric pair $(\mathfrak{g}, \mathfrak{g}_0)$. Note that $\mathfrak{c}$ is also a Cartan subalgebra of the noncompact dual of the symmetric $R$-space $G/U^-$. It is well-known that the root system $\Delta(\mathfrak{g}, \mathfrak{c})$ of $\mathfrak{g}$ with respect to $\mathfrak{c}$ is of $C_r$-type or $BC_r$-type. Correspondingly we say that the $GLA$ (1,1) and the parahermitian symmetric space $M = G/G_0$ are of $C_r$-type or $BC_r$-type, respectively. Let $\mathfrak{a}_0$ be the subspace of $\mathfrak{a}$ spanned by $\beta_1, \ldots, \beta_r$ and $\varpi$ be the orthogonal projection of $\mathfrak{a}$ onto $\mathfrak{a}_0$ with respect to $(\ ,\ )$. Then either one of the following two cases occurs ([13, 6]):

$$\begin{cases} \varpi(\Delta_1) = \left\{ \frac{1}{2}(\beta_i + \beta_j) : 1 \leq i \leq j \leq r \right\}, \\ \varpi(\Delta_0^+) - (0) = \left\{ \frac{1}{2}(\beta_i - \beta_j) : 1 \leq i < j \leq r \right\}, \end{cases} \tag{3.3}$$

$$\begin{cases} \varpi(\Delta_1) = \left\{ \frac{1}{2}(\beta_i + \beta_j)\ (1 \leq i \leq j \leq r),\ \frac{1}{2}\beta_i\ (1 \leq i \leq r) \right\}, \\ \varpi(\Delta_0^+) - (0) = \left\{ \frac{1}{2}(\beta_i - \beta_j)\ (1 \leq i < j \leq r),\ \frac{1}{2}\beta_i\ (1 \leq i \leq r) \right\}, \end{cases} \tag{3.4}$$

according as $\Delta(\mathfrak{g}, \mathfrak{c})$ is of $C_r$-type or $BC_r$-type, respectively. Here $\Delta_0^+ = \Delta_0 \cap \Delta^+$. Now let $K$ be the subgroup of $G$ consisting of elements which commute with $\tau$. Then $K$ is the maximal compact subgroup of $G$ with $\mathrm{Lie}\, K = \mathfrak{k}$. We denote the identity component of $K$ by $K^0$, and define the elements $a_l$ $(0 \leq l \leq r)$ in the normalizer $N_{K^0}(\mathfrak{a})$ of $\mathfrak{a}$ in $K^0$ by putting

$$\begin{cases} a_l = \exp -\dfrac{\pi}{2} \displaystyle\sum_{i=1}^{l} Y_i, \qquad 1 \leq l \leq r, \\ a_0 = 1. \end{cases} \tag{3.5}$$

The following theorem gives the $G$-orbit structure of $\tilde{M}$.

**Theorem 3.1.**

(i) *The points* $(0^-, a_l\, 0^+) \in \tilde{M}$, $0 \leq l \leq r$, *are a complete set of representatives of* $G$*-orbits in* $\tilde{M}$.

(ii) *Let* $M_l = G(0^-, a_{r-l}\, 0^+)$, $0 \leq l \leq r$. *Then the closure* $\overline{M_l}$ *of* $M_l$ *in* $\tilde{M}$ *is given by*

$$\overline{M_l} = M_l \amalg M_{l-1} \amalg \cdots \amalg M_0, \qquad 0 \leq l \leq r.$$

(iii) $M_r = M$ *is a single open* $G$*-orbit, and hence* $\tilde{M}$ *is a* $G$*-equivariant compactification of* $M$.

(iv) *A single closed* $G$*-orbit* $M_0$ *has the property*: *If* $M$ *is of* $C_r$*-type, then* $M_0 = M^-$ *and* $a_r\, U^+ a_r^{-1} = U^+$ *holds. If* $M$ *is of* $BC_r$*-type, then* $M_0$ *is a flag manifold of the second kind.* $M_0$ *has the double fibration*:

$$G/U^- = M^-(0^-, a_r\, 0^+) \longleftarrow M_0 \longrightarrow M^+(0^-, a_r\, 0^+) = G/a_r\, U^+ a_r^{-1}.$$

*Proof.* Let us denote by $G^0$ the identity component of $G$. In [6] we proved the theorem for $G^0$-action. But, by Theorem 4.12 in [6], $\dim M_l$ is strictly increasing, as $l$ increases. Let $g \in G$. Since $g$ normalizes $G^0$, $g(M_l)$ is still a $G^0$-orbit which has the same dimension as $M_l$. Therefore $g(M_l) = M_l$. In other words, $G$ leaves each $G^0$-orbit stable. □

## 4. Isotropy subgroups for boundary orbits

4.1. We go back to a real simple GLA $\mathfrak{g}$ in (1.1). We wish to construct a certain class of gradings of $\mathfrak{g}$ of the second kind in terms of the subsets $\Gamma_l = \{\beta_1, \dots, \beta_l\}$, $1 \le l \le r$, of $\Gamma$. When $\mathfrak{g}$ is of Hermitian type, this type of grading corresponds to the realizations of the bounded symmetric domain (corresponding to $\mathfrak{g}$) as a Siegel domain of the third kind for $1 \le l \le r-1$ and that of the second (or first) kind for $l = r$. Let $1 \le l \le r$, and put

$$
\begin{aligned}
\Delta_2(l) &= \{\, \alpha \in \Delta_1 : \varpi(\alpha) = \tfrac{1}{2}(\beta_i + \beta_j),\ 1 \le i \le j \le l \,\},\\
\Delta_1(l) &= \left\{ \alpha \in \Delta : \begin{array}{l} \varpi(\alpha) = \tfrac{1}{2}(\beta_i \pm \beta_j),\ 1 \le i \le l, l+1 \le j \le r, \text{ or} \\ \varpi(\alpha) = \tfrac{1}{2}\beta_i,\ 1 \le i \le l \end{array} \right\},\\
\Delta_0(l) &= \left\{ \alpha \in \Delta : \begin{array}{l} \varpi(\alpha) = 0, \text{ or} \\ \varpi(\alpha) = \pm\tfrac{1}{2}(\beta_i - \beta_j),\ 1 \le i < j \le l \text{ or} \\ \qquad\qquad l+1 \le i < j \le r, \text{ or} \\ \varpi(\alpha) = \pm\tfrac{1}{2}(\beta_i + \beta_j),\ l+1 \le i \le j \le r, \text{ or} \\ \varpi(\alpha) = \pm\tfrac{1}{2}\beta_i,\ l+1 \le i \le r \end{array} \right\},\\
\Delta_{-1}(l) &= -\Delta_1(l),\\
\Delta_{-2}(l) &= -\Delta_2(l).
\end{aligned}
\tag{4.1}
$$

Then, for a fixed $1 \le l \le r$, we have a partition of $\Delta$:

$$
\Delta = \coprod_{k=-2}^{2} \Delta_k(l). \tag{4.2}
$$

By using (3.3) and (3.4) we easily have

**Proposition 4.1.** *Let $1 \le l \le r$, and let $\mathfrak{c}(\mathfrak{a})$ be the centralizer of $\mathfrak{a}$ in $\mathfrak{g}$. If we put*

$$
\begin{aligned}
\mathfrak{g}_0(l) &= \mathfrak{c}(\mathfrak{a}) + \sum_{\alpha \in \Delta_0(l)} \mathfrak{g}^\alpha,\\
\mathfrak{g}_k(l) &= \sum_{\alpha \in \Delta_k(l)} \mathfrak{g}^\alpha, \qquad k = \pm 1, \pm 2,
\end{aligned}
\tag{4.3}
$$

*then we have the grading of $\mathfrak{g}$ of the second kind*

$$
\mathfrak{g} = \sum_{k=-2}^{2} \mathfrak{g}_k(l), \tag{4.4}
$$

*whose characteristic element is $Z_l = \sum_{k=1}^{l} \check{\beta}_i$.*

**Remark.** Gyoja and Yamashita [3] obtained the above gradings for $\mathfrak{g}$ complex simple, in which case there are no roots $\alpha \in \Delta$ such that $\varpi(\alpha) = 0$.

Let $s_{\beta_i}$ be the reflection on $\mathfrak{a}$ corresponding to the root $\beta_i$ $(1 \le i \le r)$, and let $s_l = s_{\beta_1} s_{\beta_2} \dots s_{\beta_l}$ $(1 \le l \le r)$ and $s_0 = 1$. It is known [13] that $\mathrm{Ad}_{\mathfrak{a}}\, a_l = s_l$, $0 \le l \le r$. Let $Q_l$ $(0 \le l \le r)$ be the isotropy subgroup of $G$ at $(0^-, a_l\, 0^+)$. Then the $G$-orbit $M_{r-l}$ can be expressed as

$$
M_{r-l} = G/Q_l, \qquad 0 \le l \le r,
$$

where $Q_l = U^- \cap a_l\, U^+ a_l^{-1}$. The Lie algebra $\mathfrak{q}_l := \operatorname{Lie} Q_l$ can be written as

$$\mathfrak{q}_l = \mathfrak{c}(\mathfrak{a}) + \sum_{\alpha\in\Psi_l} \mathfrak{g}^\alpha, \qquad 0 \le l \le r, \tag{4.5}$$

where

$$\Psi_l := \{\, \alpha \in \Delta_0 \cup \Delta_{-1} : s_l(\alpha) \in \Delta_0 \cup \Delta_1 \,\}, \qquad 0 \le l \le r. \tag{4.6}$$

**Lemma 4.2.**

$$\Psi_l = \Delta_0 \cap \Delta_0(l) \amalg \Delta_{-1}(l) \amalg \Delta_{-2}(l). \tag{4.7}$$

*Proof.* First we will show the inclusion $\supset$ in (4.7). Let $\alpha \in \Delta$. Then we have

$$\begin{aligned} \big(s_l(\alpha), Z\big) &= (\alpha, Z) - \sum_{k=1}^{l} (\alpha, \check{\beta}_k)(\beta_k, Z) \\ &= (\alpha, Z) - \sum_{k=1}^{l} 2(\varpi(\alpha), \beta_k)(\beta_k, \beta_k)^{-1}. \end{aligned} \tag{4.8}$$

Now let $\alpha \in \Delta_0 \cap \Delta_0(l)$. By using (4.8) it follows from (3.4) and (4.1) that $\big(s_l(\alpha), Z\big) = 0$, or equivalently $s_l(\alpha) \in \Delta_0$ and hence $\alpha \in \Psi_l$. Suppose next that $\alpha \in \Delta_{-1}(l)$. Then, by (4.1), there are three possibilities: $\varpi(\alpha) = -\frac{1}{2}(\beta_i + \beta_j)$ or $-\frac{1}{2}(\beta_i - \beta_j)$ for $1 \le i \le l$, $l+1 \le j \le r$, or $\varpi(\alpha) = -\frac{1}{2}\beta_i$ for $1 \le i \le l$. In view of (3.4) and (4.8), we have $\alpha \in \Delta_{-1}$ and $\big(s_l(\alpha), Z\big) = 0$ for the first case, and $\alpha \in \Delta_0$ and $\big(s_l(\alpha), Z\big) = 1$ for the second case. For the third case, there are two possibilities (cf. (3.4)): $\alpha \in \Delta_0$ or $\alpha \in \Delta_{-1}$. Then we have from (4.8) that $\big(s_l(\alpha), Z\big) = 1$ or 0, according as $\alpha \in \Delta_0$ or $\alpha \in \Delta_{-1}$, respectively. Consequently $s_l(\alpha) \in \Delta_0 \cup \Delta_1$ for $\alpha \in \Delta_{-1}(l)$. Suppose $\alpha \in \Delta_{-2}(l)$. Then by (3.4) and (4.8) we have $\alpha \in \Delta_{-1}$ and $\big(s_l(\alpha), Z\big) = 1$.

To prove the converse inclusion $\subset$ in (4.7), let $\alpha \in \Psi_l$ and suppose that $\alpha$ does not belong to the right-hand side of (4.7). Then the following three cases occur:

(i) $\alpha \in \Delta_2(l)$,
(ii) $\alpha \in \Delta_1(l)$ and
(iii) $\alpha \in \Delta_0(l) - \Delta_0$.

For (i), we have $\alpha \in \Delta_1$, contradicting the assumption of $\alpha \in \Psi_l$. For (ii), we have three possibilities: $\varpi(\alpha) = \frac{1}{2}(\beta_i + \beta_j)$ or $\frac{1}{2}(\beta_i - \beta_j)$ both for $1 \le i \le l$, $l+1 \le j \le r$ or $\varpi(\alpha) = \frac{1}{2}\beta_i$ for $1 \le i \le l$. For the first case we have $\alpha \in \Delta_1$, which contradicts $\alpha \in \Psi_l$. For the second case, we have $\alpha \in \Delta_0$. Consequently, by using (4.8), we have that $\big(s_l(\alpha), Z\big) = -1$, that is, $s_l(\alpha) \in \Delta_{-1}$. This contradicts the assumption $\alpha \in \Psi_l$. For the third case, we have either $\alpha \in \Delta_1$ or $\alpha \in \Delta_0$. In view of the condition $\alpha \in \Psi_l$, we have the only choice $\alpha \in \Delta_0$, in which case $\big(s_l(\alpha), Z\big) = -1$, still contradicting the assumption $\alpha \in \Psi_l$. Let us consider the case (iii) finally. Since $\alpha$ lies in $\Psi_l \cap \big(\Delta_0(l) - \Delta_0\big)$, we have that $\varpi(\alpha) = -\frac{1}{2}(\beta_i + \beta_j)$, $l+1 \le i \le j \le r$, or $\varpi(\alpha) = -\frac{1}{2}\beta_i$, $l+1 \le i \le r$. In particular $\alpha \in \Delta_{-1}$. Therefore, in both cases, we have $\big(s_l(\alpha), Z\big) = -1$, which contradicts the assumption $\alpha \in \Psi_l$. □

**Proposition 4.3.** *The isotropy subalgebra $\mathfrak{q}_l$ of $\mathfrak{g}$ at the point $(0^-, a_l\, 0^+)$ is given by*

$$\mathfrak{q}_l = \mathfrak{g}_{-2}(l) + \mathfrak{g}_{-1}(l) + \mathfrak{g}_0(l) \cap \mathfrak{g}_0, \qquad 0 \le l \le r.$$

*Proof.* This follows immediately from Lemma 4.2 and (4.5). □

4.2. In this paragraph we will determine the isotropy subgroup $Q_r$ of $G$ at the point $(0^-, a_r\, 0^+) \in M_0$. Let us consider the special case $l = r$. Then (4.1) has the following simple form:

$$\begin{aligned}
\Delta_2(r) &= \big\{\, \alpha \in \Delta_1 : \varpi(\alpha) = \tfrac{1}{2}(\beta_i + \beta_j),\ 1 \le i \le j \le r \,\big\},\\
\Delta_1(r) &= \big\{\, \alpha \in \Delta_0^+ \cup \Delta_1 : \varpi(\alpha) = \tfrac{1}{2}\beta_i,\ 1 \le i \le r \,\big\},\\
\Delta_0(r) &= \big\{\, \alpha \in \Delta : \varpi(\alpha) = 0 \text{ or } = \pm\tfrac{1}{2}(\beta_i - \beta_j),\ 1 \le i < j \le r \,\big\},\\
\Delta_{-k}(r) &= -\Delta_k(r), \qquad k = 1, 2.
\end{aligned} \tag{4.9}$$

We have the grading of the second kind

$$\mathfrak{g} = \sum_{k=-2}^{2} \mathfrak{g}_k(r). \tag{4.10}$$

**Remark.** In the case of $C_r$-type, we have $\Delta_1(r) = \emptyset$, $\Delta_2(r) = \Delta_1$ and $\Delta_0(r) = \Delta_0$. Hence the grading (4.10) is reduced to the grading (1.1).

**Lemma 4.4.** $\mathfrak{q}_r = \mathfrak{g}_{-2}(r) + \mathfrak{g}_{-1}(r) + \mathfrak{g}_0(r)$. *In particular,* $\mathfrak{q}_r$ *is a parabolic subalgebra of* $\mathfrak{g}$ *of the second kind.*

*Proof.* Since $\Delta_0(r) \subset \Delta_0$, we have the inclusion $\mathfrak{g}_0(r) \subset \mathfrak{g}_0$. □

Now we put

$$\begin{aligned}
\Delta_1^+(r) &= \Delta_1(r) \cap \Delta_0^+, & \Delta_1^-(r) &= \Delta_1(r) \cap \Delta_1,\\
\Delta_{-1}^+(r) &= \Delta_{-1}(r) \cap \Delta_{-1}, & \Delta_{-1}^-(r) &= \Delta_{-1}(r) \cap \Delta_0^-.
\end{aligned} \tag{4.11}$$

Then we have

$$\Delta_{\pm1}(r) = \Delta_{\pm1}^+(r) \amalg \Delta_{\pm1}^-(r). \tag{4.12}$$

We define the four subspaces of $\mathfrak{g}$:

$$\mathfrak{g}_{\pm1}^+(r) = \sum_{\alpha \in \Delta_{\pm1}^+(r)} \mathfrak{g}^\alpha, \qquad \mathfrak{g}_{\pm1}^-(r) = \sum_{\alpha \in \Delta_{\pm1}^-(r)} \mathfrak{g}^\alpha. \tag{4.13}$$

Those four subspaces are equi-dimensional and abelian ([6]). We have the decompositions

$$\mathfrak{g}_1(r) = \mathfrak{g}_1^-(r) + \mathfrak{g}_1^+(r), \qquad \mathfrak{g}_{-1}(r) = \mathfrak{g}_{-1}^+(r) + \mathfrak{g}_{-1}^-(r). \tag{4.14}$$

The original grading (1.1) of $\mathfrak{g}$ can be reconstructed as

$$\begin{aligned}
\mathfrak{g}_{-1} &= \mathfrak{g}_{-2}(r) + \mathfrak{g}_{-1}^+(r),\\
\mathfrak{g}_0 &= \mathfrak{g}_{-1}^-(r) + \mathfrak{g}_0(r) + \mathfrak{g}_1^+(r),\\
\mathfrak{g}_1 &= \mathfrak{g}_1^-(r) + \mathfrak{g}_2(r).
\end{aligned} \tag{4.15}$$

Let $C(Z_r)$ be the centralizer of $Z_r$ in $\operatorname{Aut}\mathfrak{g}$. Then the normalizer $N(\mathfrak{q}_r)$ in $\operatorname{Aut}\mathfrak{g}$ of $\mathfrak{q}_r$ can be written as

$$N(\mathfrak{q}_r) = C(Z_r) \cdot \exp\big(\mathfrak{g}_{-2}(r) + \mathfrak{g}_{-1}(r)\big). \qquad \text{(semi-direct)} \tag{4.16}$$

$Q_r$ is a subgroup of $U^- \cap N(\mathfrak{q}_r) = N_{U^-}(\mathfrak{q}_r)$, the normalizer of $\mathfrak{q}_r$ in $U^-$. (4.15) implies that

$$N_{U^-}(\mathfrak{q}_r) = \big(C(Z_r) \cap U^-\big) \cdot \exp\big(\mathfrak{g}_{-2}(r) + \mathfrak{g}_{-1}(r)\big). \tag{4.17}$$

**Lemma 4.5.** *Let* $C(Z, Z_r)$ *be the centralizer of both elements* $Z$ *and* $Z_r$ *in* $\operatorname{Aut}\mathfrak{g}$. *Then we have* $C(Z_r) \cap U^- = C(Z, Z_r)$.

*Proof.* Let $a \in C(Z_r) \cap U^-$. We write $a = b \exp X,\ b \in C(Z),\ X \in \mathfrak{g}_{-1}$. Then $[X, Z_r] \in \mathfrak{g}_{-1}$ and hence $\big[X, [X, Z_r]\big] = 0$ Therefore we have

$$Z_r = (\operatorname{Ad} a)Z_r = (\operatorname{Ad} b)(\operatorname{Ad} \exp X)Z_r = (\operatorname{Ad} b)Z_r + (\operatorname{Ad} b)[X, Z_r] \subset \mathfrak{g}_0 + \mathfrak{g}_{-1}.$$

Hence $(\operatorname{Ad} b)[X, Z_r] = 0$. $\operatorname{Ad} b$ being invertible on $\mathfrak{g}_{-1}$, we have $[X, Z_r] = 0$. Consequently $X \in \mathfrak{g}_0(r) \cap \mathfrak{g}_{-1} = (0)$, and $a = b \in C(Z)$. □

**Proposition 4.6.** *The isotropy subgroup $Q_r$ of $G$ at $(0^-, a_r\, 0^+) \in M_0$ is given by*

$$Q_r = C(Z, Z_r) \exp\big(\mathfrak{g}_{-2}(r) + \mathfrak{g}_{-1}(r)\big).$$

*Proof.* By Lemma 4.5 and (4.17) we have

$$N_{U^-}(\mathfrak{q}_r) = C(Z, Z_r) \exp\big(\mathfrak{g}_{-2}(r) + \mathfrak{g}_{-1}(r)\big).$$

Recall $Q_r \subset N_{U^-}(\mathfrak{q}_r)$. To prove the converse inclusion, it suffices to show that $C(Z, Z_r) \subset Q_r$. By using (4.8), one sees $s_r(Z) = Z - Z_r$, and consequently

$$a_r^{-1} C(Z) a_r = C\big((\operatorname{Ad} a_r^{-1})Z\big) = C\big(s_r(Z)\big) = C(Z - Z_r),$$
$$a_r^{-1} C(Z_r) a_r = C\big((\operatorname{Ad} a_r^{-1})Z_r\big) = C\big(s_r(Z_r)\big) = C(Z_r).$$

As a result, $a_r^{-1} C(Z, Z_r) a_r = C(Z - Z_r) \cap C(Z_r) = C(Z, Z_r) \subset U^+$. Hence we have $C(Z, Z_r) \subset U^- \cap a_r\, U^+ a_r^{-1} = Q_r$. □

**Corollary 4.7.** $G = C(Z, Z_r) G^0$.

*Proof.* We have $M_0 = G/Q_r = G^0/Q_r \cap G^0 = G^0 Q_r/Q_r$, which implies $G = Q_r G^0 = C(Z, Z_r) G^0$. □

**Corollary 4.8.** *Suppose that $M$ is of $C_r$-type. Then $Q_r = U^- = a_r\, U^+ a_r^{-1}$ and $M_0 = G/U^- = M^-$.*

*Proof.* By the Remark before Lemma 4.4 and Proposition 4.6, we see that $Q_r = C(Z) \exp \mathfrak{g}_{-1}$ $= U^-$, and hence $U^- = Q_r = U^- \cap a_r\, U^+ a_r^{-1} \subset a_r\, U^+ a_r^{-1}$. Hence we have $U^- = a_r\, U^+ a_r^{-1}$. □

## 5. Siegel-type realization of orbits

5.1.

**Lemma 5.1.**

$$s_r\big(\Delta_k(r)\big) = \Delta_{-k}(r), \qquad k = 0, \pm 1, \pm 2, \tag{5.1}$$
$$s_r\big(\Delta_{-1}^{\pm}(r)\big) = \Delta_1^{\pm}(r). \tag{5.2}$$

*Proof.* Let $\alpha \in \Delta_k(r)$. Then $\big(s_r(\alpha), Z_r\big) = \big(\alpha, s_r(Z_r)\big) = -(\alpha, Z_r) = -k$, which implies that $s_r(\alpha) \in \Delta_{-k}(r)$. Let $\alpha \in \Delta_{-1}^+(r)$. Then, by (4.8) we have $\big(s_r(\alpha), Z\big) = 0$, and hence $s_r(\alpha) \in \Delta_0$. One can write $\varpi(\alpha) = -\frac{1}{2}\beta_i$ for some $i$. Hence we have $\varpi\big(s_r(\alpha)\big) = s_r \varpi(\alpha) = s_r(-\frac{1}{2}\beta_i) = \frac{1}{2}\beta_i$, proving that $s_r(\alpha) \in \Delta_1^+(r)$. □

**Lemma 5.2.** *The operator $\operatorname{Ad} a_r$ is grade-reversing with respect to the grading* (4.10). *Moreover $\operatorname{Ad} a_r$ interchanges $\mathfrak{g}_{-1}^{\pm}(r)$ with $\mathfrak{g}_1^{\pm}(r)$, respectively.*

*Proof.* Since $\operatorname{Ad} a_r$ induces $s_r$ on $\mathfrak{a}$ (cf. 4.1), the lemma is immediate from Lemma 5.1. □

5.2. Up to the present, we have expressed $\tilde{M}$ as $M^- \times M^+$. Here $M^\pm$ are just the leaves of the product foliation through the origin $(0^-, 0^+)$. In order to get the Siegel-type realization of $G$-orbits, we choose the point $(0^-, a_r\, 0^+) \in M_0$ as the new origin of $\tilde{M}$. Then $\tilde{M}$ can be expressed as

$$\tilde{M} = M^-(0^-, a_r\, 0^+) \times M^+(0^-, a_r\, 0^+) = G/U^- \times G/a_r\, U^+ a_r^{-1}. \tag{5.3}$$

For simplicity we write $(M^+)_r$ for $G/a_r\, U^+ a_r^{-1}$. We identify the tangent space $T_{0^-}(G/U^-)$ with $\mathfrak{g}_1 = \mathfrak{g}_2(r) + \mathfrak{g}_1^-(r)$ (cf. (4.15)), and $T_{0^+}(G/U^+)$ with $\mathfrak{g}_{-1} = \mathfrak{g}_{-2}(r) + \mathfrak{g}_{-1}^+(r)$. Then the tangent space $T_{a_r\, 0^+}(G/a_r\, U^+ a_r^{-1})$ can be identified with $(\operatorname{Ad} a_r)\mathfrak{g}_{-1} = \mathfrak{g}_2(r) + \mathfrak{g}_1^+(r)$ by Lemma 5.2. We will denote $(\operatorname{Ad} a_r)\mathfrak{g}_{-1}$ by $\mathfrak{g}_1'$. Let us consider the direct sum of the vector space

$$\mathfrak{g}_1 \oplus \mathfrak{g}_1' = \big(\mathfrak{g}_2(r) + \mathfrak{g}_1^-(r)\big) \oplus \big(\mathfrak{g}_2(r) + \mathfrak{g}_1^+(r)\big). \tag{5.4}$$

We define the map $\xi$ of $\mathfrak{g}_1 \oplus \mathfrak{g}_1'$ into $\tilde{M} = M^- \times (M^+)_r$ by

$$\xi(X, X') = \big((\exp X)0^-, (\exp X')a_r\, 0^+\big). \tag{5.5}$$

Then $\xi$ is an open dense imbedding of $\mathfrak{g}_1 \oplus \mathfrak{g}_1'$. We will always identify $\mathfrak{g}_1 \oplus \mathfrak{g}_1'$ with its $\xi$-image.

**Lemma 5.3.** *Let* $a_l^\pm = \exp \sum_{i=1}^l E_{\pm i}$ $(1 \le l \le r)$. *Then*

$$a_r^{-1}(a_l^-)^{-1} a_r = a_l^+, \qquad 1 \le l \le r. \tag{5.6}$$

*Proof.* Consider the elements in $\mathfrak{sl}(2, \mathbb{R})$

$$e_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad e_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

By easy computation we have

$$\exp \frac{\pi}{2}(e_+ - e_-) \exp(-e_-) \exp -\frac{\pi}{2}(e_+ - e_-) = \exp e_+.$$

Let $\varphi_i \colon \mathfrak{sl}(2, \mathbb{R}) \to \mathfrak{g}$ $(1 \le i \le r)$ be the maps defined by $\varphi_i(e_\pm) = E_{\pm i}$. By the strong orthogonality of the $\beta_i$, we have $[\varphi_i, \varphi_j] = 0$ $(i \ne j)$. $\varphi_i$ can be extended to the homomorphism of $\mathrm{SL}(2, \mathbb{R})$ to $G$, denoted again be $\varphi_i$.

LHS of (5.6)

$$\begin{aligned}
&= \exp \frac{\pi}{2} \sum_{i=1}^r \big(\varphi_i(e_+) - \varphi_i(e_-)\big) \exp\Big(-\sum_{i=1}^l \varphi_i(e_-)\Big) \exp\Big(-\frac{\pi}{2} \sum_{i=1}^r \big(\varphi_i(e_+) - \varphi_i(e_-)\big)\Big) \\
&= \prod_{i=1}^r \varphi_i\Big(\exp \frac{\pi}{2}(e_+ - e_-)\Big) \prod_{i=1}^l \varphi_i(\exp -e_-) \prod_{i=1}^r \varphi_i\Big(\exp\Big(-\frac{\pi}{2}(e_+ - e_-)\Big)\Big) \\
&= \prod_{i=1}^l \varphi_i\Big(\exp \frac{\pi}{2}(e_+ - e_-) \exp(-e_-) \exp\Big(-\frac{\pi}{2}(e_+ - e_-)\Big)\Big) \\
&= \prod_{i=1}^l \varphi_i(\exp e_+) = \exp \sum_{i=1}^l \varphi_i(e_+) = \exp \sum_{i=1}^l E_i.
\end{aligned}$$

□

The following lemma was proved in [9]. Note that we do not use there the assumption that $\Delta(\mathfrak{g}, \mathfrak{c})$ is of $C_r$-type.

**Lemma 5.4.** $a_l^- a_l^{-1} a_l^- = a_l^+$.

**Lemma 5.5.** $(0^-, a_l a_r\, 0^+) \equiv (a_l^+\, 0^-, (a_l^+)^2 a_r\, 0^+) \bmod G$.

*Proof.* First note that $a_l\, 0^\pm = a_l^{-1}\, 0^\pm$. In fact, $\operatorname{Ad} a_l^2$ is the identity on $\mathfrak{a}$, which implies that $a_l^2$ lies in the centralizer $C(Z) = U^+ \cap U^-$. Also note that $a_l^{-1} a_r = a_r a_l^{-1}$, since $a_l$ and $a_r$ commute. Consequently, in view of Lemmas 5.4 and 5.3, we have

$$\begin{aligned}(0^-, a_l a_r\, 0^+) &= (0^-, a_r a_l\, 0^+) = (0^-, a_r a_l^{-1}\, 0^+) = (0^-, a_l^{-1} a_r\, 0^+)\\ &\equiv (a_l^+ a_l^-\, 0^-, a_l^+ a_l^- a_l^{-1} a_r\, 0^+) = (a_l^+\, 0^-, (a_l^+)^2 (a_l^-)^{-1} a_r\, 0^+)\\ &= (a_l^+\, 0^-, (a_l^+)^2 a_r a_l^+\, 0^+) = (a_l^+\, 0^-, (a_l^+)^2 a_r\, 0^+) \bmod G. \qquad \square\end{aligned}$$

Let us put $0_l = \sum_{i=1}^{l} E_i \in \mathfrak{g}_2(r)$, $1 \le l \le r$, $0_0 = 1$.

**Proposition 5.6.** *The point $(0_l, 2\,0_l) \in \mathfrak{g}_1 \oplus \mathfrak{g}_1'$ (identified with its $\xi$-image) is a representative of the $G$-orbit $M_l$ for $0 \le l \le r$.*

*Proof.*

$$\begin{aligned}M_l &= G(0^-, a_{r-l}\, 0^+) = G(0^-, a_l a_r\, 0^+) = G(a_l^+\, 0^-, (a_l^+)^2 a_r\, 0^+)\\ &= G\big((\exp 0_l)\, 0^-, (\exp 0_l)^2 a_r\, 0^+\big) = G\big((\exp 0_l)\, 0^-, (\exp 2\,0_l) a_r\, 0^+\big)\\ &= G\big(\xi(0_l, 2\,0_l)\big). \qquad \square\end{aligned}$$

5.3. A preliminary step for Siegel-type realization of orbits was done by Tanaka [15], which is needed for later consideration. Let $\widehat{Q_r}$ be the parabolic subgroup of $G$ opposite to $Q_r$, that is, $\widehat{Q_r} = C(Z, Z_r) \cdot N$, where $N = \exp \mathfrak{n}$ and $\mathfrak{n} = \mathfrak{g}_2(r) + \mathfrak{g}_1(r) = \mathfrak{g}_2(r) + \mathfrak{g}_1^+(r) + \mathfrak{g}_1^-(r)$. The group $G$ acts on the vector space $\mathfrak{g}_1 \oplus \mathfrak{g}_1'$ birationally through $\xi$. But the subgroup $\widehat{Q}$ acts on it as affine transformations.

**Proposition 5.7** (Tanaka [15])**.** *Let $a$, $x$, $y \in \mathfrak{g}_2(r)$, $b^+$, $v^+ \in \mathfrak{g}_1^+(r)$, $b^-$, $u^- \in \mathfrak{g}_1^-(r)$ and $h \in C(Z, Z_r)$. Then the $\xi$-equivariant action of $\widehat{Q_r}$ is given by*

$$\begin{aligned}&\exp(a + b^+ + b^-)\big((x, u^-) \oplus (y, v^+)\big)\\ &\quad = (x + a + [b^+, u^-] + \tfrac{1}{2}[b^+, b^-], u^- + b^-)\\ &\qquad\qquad \oplus (y + a + [b^-, v^+] + \tfrac{1}{2}[b^-, b^+], v^+ + b^+),\end{aligned} \tag{5.7}$$

$$\begin{aligned}&h\big((x, u^-) \oplus (y, v^+)\big)\\ &\quad = \big((\mathrm{Ad}_{\mathfrak{g}_2(r)}\, h)x, (\mathrm{Ad}_{\mathfrak{g}_1^-(r)}\, h)u^-\big) \oplus \big((\mathrm{Ad}_{\mathfrak{g}_2(r)}\, h)y, (\mathrm{Ad}_{\mathfrak{g}_1^+(r)}\, h)v^+\big).\end{aligned} \tag{5.8}$$

**Definition 5.1.** We define a surjective submersion $\Phi$ of $\mathfrak{g}_1 \oplus \mathfrak{g}_1'$ onto $\mathfrak{g}_2(r)$ as follows: For $X = (x, u^-) \in \mathfrak{g}_1$, $Y = (y, v^+) \in \mathfrak{g}_1'$,

$$\Phi(X \oplus Y) = y - x + [v^+, u^-].$$

$\Phi$ has the following $\big(\widehat{Q_r}, C(Z, Z_r)\big)$-equivariancy property.

**Proposition 5.8** ([15])**.** *$\Phi$ is invariant under the action of $N$. Moreover let $h \in C(Z, Z_r)$ and let $X' \oplus Y' = h(X \oplus Y)$. Then*

$$\Phi(X' \oplus Y') = (\mathrm{Ad}_{\mathfrak{g}_2(r)}\, h)\Phi(X \oplus Y).$$

We restate Lemma 3.8 [15] as follows:

**Lemma 5.9.** *$N$ acts on $\mathfrak{g}_1 \oplus \mathfrak{g}_1'$ freely. Moreover let $X \oplus Y \in \mathfrak{g}_1 \oplus \mathfrak{g}_1'$ and let $X = (x, u^-)$, $Y = (y, v^+)$. Then we have*

$$\exp(-v^+)\exp(-x-u^-)(X \oplus Y) = (0,0) \oplus \big(\Phi(X,Y), 0\big). \tag{5.9}$$

Note that the group $C(Z_r)$ is the group of grade-preserving automorphisms with respect to the grading (4.10) and $\mathrm{Lie}\, C(Z_r) = \mathfrak{g}_0(r)$. $C(Z, Z_r)$ is an open subgroup of $C(Z_r)$.

Let $\widehat{Q_r}^0$ and $C^0(Z_r)$ be the identity components of $\widehat{Q_r}$ and $C(Z_r)$, respectively. The following proposition follows from Proposition 5.8 (cf.[15]).

**Proposition 5.10.** *There exists a bijection between the set of $\widehat{Q_r}^0$-orbits in $\mathfrak{g}_1 \oplus \mathfrak{g}_1'$ and the set of $C^0(Z_r)$-orbits in $\mathfrak{g}_2(r)$. More precisely, the $\Phi$-image of a $\widehat{Q_r}^0$-orbit is a $C^0(Z_r)$-orbit, and the complete inverse image by $\Phi$ of a $C^0(Z_r)$-orbit is a $\widehat{Q_r}^0$-orbit.*

5.4. We are interested in the intersection of a $G$-orbit with $\mathfrak{g}_1 \oplus \mathfrak{g}_1'$. Let $M_l^* = M_l \cap (\mathfrak{g}_1 \oplus \mathfrak{g}_1')$, $0 \le l \le r$, which is a dense open set in $M_l$. $M_l^*$ is stable under $\widehat{Q_r}$, and hence it can be expressed as the union of $\widehat{Q_r}^0$-orbits contained in $M_l^*$. Those $\widehat{Q_r}^0$-orbits are open in $M_l^*$ ([15]). This fact can also be proved by using Proposition 4.3. Let us consider the (reductive) graded subalgebra of $\mathfrak{g}$

$$\mathfrak{g}_{\mathrm{ev}}(r) = \mathfrak{g}_{-2}(r) + \mathfrak{g}_0(r) + \mathfrak{g}_2(r), \tag{5.10}$$

which contains the simple graded ideal

$$\mathfrak{g}'_{\mathrm{ev}}(r) = \mathfrak{g}_{-2}(r) + [\mathfrak{g}_{-2}(r), \mathfrak{g}_2(r)] + \mathfrak{g}_2(r). \tag{5.11}$$

By the table of $(\mathfrak{g}, \mathfrak{g}_{\mathrm{ev}})$ in [7], it turns out that $\mathfrak{g}_2(r)$ has the structure of a real simple Jordan algebra and the adjoint action of $C^0(Z_r)$ on $\mathfrak{g}_2(r)$ coincides with the identity component of the structure group of this Jordan algebra. Therefore we have the rank decomposition ([2])

$$\mathfrak{g}_2(r) = V_r \amalg V_{r-1} \amalg \cdots \amalg V_0, \tag{5.12}$$

where $V_l$ is the union of equi-dimensional $C^0(Z_r)$-orbits. $V_r$ is open dense in $\mathfrak{g}_2(r)$, $\dim V_k > \dim V_{k-1}$, and $V_0 = (0)$.

**Lemma 5.11.** *Let $p = X \oplus Y \in \mathfrak{g}_1 \oplus \mathfrak{g}_1'$. Then $\Phi^{-1}\big(\Phi(p)\big)$ is the $N$-orbit through the point $p$.*

*Proof.* This is an easy consequence of Proposition 5.8 and Lemma 5.9. □

**Proposition 5.12.** *$M_l^* = \Phi^{-1}(V_l)$, $0 \le l \le r$.*

*Proof.* By Proposition 5.6, we have $M_l^* = M_l \cap (\mathfrak{g}_1 \oplus \mathfrak{g}_1') = G^0(0_l, 2\,0_l) \cap (\mathfrak{g}_1 \oplus \mathfrak{g}_1')$, which contains the same dimensional orbit $\widehat{Q_r}^0(0_l, 2\,0_l)$. On the other hand

$$\widehat{Q_r}^0(0_l, 2\,0_l) = \Phi^{-1}\big(C^0(Z_r)\Phi(0_l, 2\,0_l)\big) = \Phi^{-1}\big(C^0(Z_r)\,0_l\big).$$

Let

$$0_{p,q} = \sum_{i=1}^{p} E_i - \sum_{j=p+1}^{p+q} E_j.$$

Note that $0_l = 0_{l,0}$. Let $V_{p,q} = C^0(Z_r)\,0_{p,q}$. It is known [8] that $V_l = \coprod_{p+q=l} V_{p,q}$. Those spaces $V_{p,q}$ in the right-hand side exhaust all $C^0(Z_r)$-orbits of the dimension equal to

$\dim C^0(Z_r)\,0_l$. By Lemma 5.11, $\Phi^{-1}(V_{p,q})$, $p+q=l$, are the same dimensional $\widehat{Q_r}^0$-orbits. Therefore, by Proposition 5.10, we have

$$\Phi^{-1}(V_l) = \Phi^{-1}\Big(\coprod_{p+q=l} V_{p,q}\Big) = \coprod_{p+q=l} \Phi^{-1}(V_{p,q}) = M_l^*. \qquad \square$$

We say that $\Phi^{-1}(V_l)$ is the Siegel-type realization of the $G$-orbit $M_l$. Let $P\colon \mathfrak{g}_2(r) \to \operatorname{End}\mathfrak{g}_2(r)$ be the quadratic operator of the Jordan algebra $\mathfrak{g}_2(r)$. Then we have

**Theorem 5.13.** *The Siegel-type realization of the $G$-orbit $M_l$, $0 \le l \le r$, is given by*

$$M_l^* = \Phi^{-1}(V_l) = \{\,(x,u^-)\oplus(y,v^+) \in \mathfrak{g}_1\oplus\mathfrak{g}_1' : \operatorname{rk} P(y-x+[v^+,u^-]) = i_l\,\}, \tag{5.13}$$

*where $i_l = \operatorname{rk} P(0_l)$. In particular, when $l=r$, the Siegel-type realization of the parahermitian symmetric space $M = G/G_0$ is given by*

$$M_r^* = \Phi^{-1}(V_r) = \{\,(x,u^-)\oplus(y,v^+) \in \mathfrak{g}_1\oplus\mathfrak{g}_1' : \nu(y-x+[v^+,u^-]) \neq 0\,\}, \tag{5.14}$$

*where $\nu$ denotes the generic norm of the Jordan algebra $\mathfrak{g}_2(r)$.*

*Proof.* By Proposition 5.12, we have

$$M_l^* = \Phi^{-1}(V_l) = \{\,(x,u^-)\oplus(y,v^+) \in \mathfrak{g}_1\oplus\mathfrak{g}_1' : y-x+[v^+,u^-] \in V_l\,\}. \tag{5.15}$$

Also we have [2] that

$$V_l = \{\,x \in \mathfrak{g}_2(r) : \operatorname{rk} P(x) = i_l\,\}. \tag{5.16}$$

Note that the condition $\operatorname{rk} P(x) = i_r$ is equivalent to the condition $\nu(x) \neq 0$. $\square$

**Remark.** (5.14) is an analogue of the Siegel domain realization of a bounded symmetric domain. In the case of $C_r$-type, (5.13) was obtained in [9], in which case $v^+ = u^- = 0$.

The closure $\overline{V_l}$ of $V_l$ in $\mathfrak{g}_2(r)$, was given by $V_{\le l} := \coprod_{i=0}^{l} V_i$ ([2]). Therefore, from (5.16) it follows that $\overline{V_l} = V_{\le l}$ is an algebraic variety in $\mathfrak{g}_2(r)$, which we call a *generalized determinantal variety*. In the case of $\mathfrak{g}_2(r) = M_n(\mathbb{C})$, $\operatorname{Sym}_n(\mathbb{C})$ (resp. $\operatorname{Alt}_{2n}(\mathbb{C})$), the number $l$ is just the rank (resp. one-half of the rank) of a matrix for $M_n(\mathbb{C})$ and $\operatorname{Sym}_n(\mathbb{C})$ (resp. $\operatorname{Alt}_{2n}(\mathbb{C})$). In those cases, $V_{\le l}$ is a usual determinantal variety.

## 6. Stratifications of $\tilde{M}$

6.1. We wish to construct a polynomial map $\Psi$ of $\mathfrak{g}_1\oplus\mathfrak{g}_1'$ to $\mathfrak{g}_2(r)\times\mathfrak{n}$ (cf.5.3). Choose a point $p = (x,u^-)\oplus(y,v^+) \in \mathfrak{g}_1\oplus\mathfrak{g}_1'$, and consider the element $n_p = \exp(-v^+)\exp(-x-u^-) \in N$. $n_p$ can be written as $n_p = \exp(-x+\frac{1}{2}[v^+,u^-]-v^+-u^-)$. Since $\exp : \mathfrak{n} \to N$ is diffeomorphic, one can define $\Psi$ to be

$$\Psi(p) = \big(\Phi(p), \log n_p\big) = \Big(y-x+[v^+,u^-], -x+\frac{1}{2}[v^+,u^-]-v^+-u^-\Big). \tag{6.1}$$

**Lemma 6.1.** *The polynomial map $\Psi$ is a diffeomorphism of $\mathfrak{g}_1\oplus\mathfrak{g}_1'$ onto $\mathfrak{g}_2(r)\times\mathfrak{n}$. $\Psi^{-1}$ is also a polynomial map.*

*Proof.* Let $p, q \in \mathfrak{g}_1\oplus\mathfrak{g}_1'$, and suppose $\Psi(p) = \Psi(q)$. Then we have $\Phi(p) = \Phi(q)$ and $n_p = n_q$. By Lemma 5.9, we have that $n_p(p) = (0,0)\oplus\big(\Phi(p),0\big) = (0,0)\oplus\big(\Phi(q),0\big) = n_q(q) = n_p(q)$, which implies that $p = q$, proving the injectivity of $\Psi$. Now let $(a,X) \in \mathfrak{g}_2(r)\times\mathfrak{n}$, and let $p := (\exp\,-X)\big((0,0)\oplus(a,0)\big) \in \mathfrak{g}_1\oplus\mathfrak{g}_1'$. Then, by Lemma 5.11, $\Phi(p) = \Phi\big((0,0)\oplus(a,0)\big) = a$, and hence, by Lemma 5.9, we have $(\exp\,X)p = (0,0)\oplus\big(\Phi(p),0\big) = n_p(p)$. Since $N$ acts

freely on $\mathfrak{g}_1 \oplus \mathfrak{g}_1'$, it follows that $(a, X) = (\Phi(p), \log n_p)$, proving the surjectivity of $\Psi$. On the other hand, $\Psi^{-1}$ is given by $\Psi^{-1}(a, X) = (\exp -X)\big((0,0) \oplus (a,0)\big)$, which is a polynomial in $a$ and $X$ by (5.7). □

From the expression of $\Psi^{-1}$ in the above proof, we have

**Lemma 6.2.** *Let $V$ be a $C^0(Z_r)$-orbit in $\mathfrak{g}_2(r)$. Then, for the corresponding $\widehat{Q_r}^0$-orbit $\Phi^{-1}(V)$, we have $\Psi(\Phi^{-1}(V)) = V \times \mathfrak{n}$.*

Let $M_{\leq k}$ and $M^*_{\leq k}$ denote the unions $\coprod_{i=0}^k M_i$ and $\coprod_{i=0}^k M_i^*$, respectively. By Theorem 3.1 and Proposition 5.12, the closure $\overline{M_k^*}$ of $M_k^*$ in $\mathfrak{g}_1 \oplus \mathfrak{g}_1'$ is given by $\overline{M_k^*} = M^*_{\leq k} = \Phi^{-1}(V_{\leq k})$, which is an algebraic variety in $\mathfrak{g}_1 \oplus \mathfrak{g}_1'$ (cf. Theorem 5.13). Since $V_{\leq k}$ is an algebraic variety in $\mathfrak{g}_2(r)$ ([2]), $V_{\leq k} \times \mathfrak{n}$ is an algebraic variety in $\mathfrak{g}_2(r) \times \mathfrak{n}$. We will denote the singular locus and the regular locus of an algebraic variety $A$ by $\mathrm{Sing}(A)$ and $\mathrm{Reg}(A)$, respectively.

**Proposition 6.3.** *The algebraic variety $M^*_{\leq k}$ is isomorphic to the algebraic variety $V_{\leq k} \times \mathfrak{n}$, for $0 \leq k \leq r-1$. We have*

$$\Psi(\mathrm{Sing}(M^*_{\leq k})) = \mathrm{Sing}(V_{\leq k}) \times \mathfrak{n},\ 0 \leq k \leq r-1.$$

*In other words,*

$$\mathrm{Sing}(M^*_{\leq k}) = \Phi^{-1}(\mathrm{Sing}(V_{\leq k})),\ 0 \leq k \leq r-1.$$

*Proof.* The first assertion is an immediate consequence of Lemmas 6.1 and 6.2. The other assertions follow from the first one. □

6.2. We wish to find the singular locus $\mathrm{Sing}(V_{\leq k})$ of a generalized determinantal variety $V_{\leq k}$ in $\mathfrak{g}_2(r)$. In the case where $(\mathfrak{g}, \mathfrak{g}_0)$ is of $BC_r$-type, $V_{\leq k}$ is determined by the graded subalgebra (5.10) or (5.11) of the first kind. As is seen from the classification of simple GLA's of the 2nd kind([8]), the simple GLA (5.11) is of $C_r$-type in this case. As for the case where $(\mathfrak{g}, \mathfrak{g}_0)$ is of $C_r$-type, the GLA (5.11) coincides with the original GLA (1.1), more precisely, we have $\mathfrak{g}_{\pm 2}(r) = \mathfrak{g}_{\pm 1}$ and $\mathfrak{g}_0(r) = \mathfrak{g}_0$. Therefore one has only to consider the generalized determinantal varieties arising from a simple GLA (1.1) of $C_r$-type.

Consider a simple GLA (1.1) of $C_r$-type. In this case $\mathfrak{g}_1$ is a simple Jordan algebra on which $G_0$ acts as the structure group. As for (5.12) we have the rank decomposition

$$\mathfrak{g}_1 = V_r \amalg V_{r-1} \amalg \cdots \amalg V_0, \tag{6.2}$$

where $V_r$ is an open subset and $V_0 = (0)$. If we denote by $G_0^0$ the identity component of $G_0$, then $V_k$ is a union of the equidimensional orbits [8]:

$$V_k = \amalg_{p+q=k} G_0^0 0_{p,q},\ 0 \leq k \leq r, \tag{6.3}$$

where $0_{p,q}$ is the same as in the proof of Proposition 5.12. (5.16) is still valid by replacing $\mathfrak{g}_2(r)$ by $\mathfrak{g}_1$. Therefore $V_{\leq k}$ is an algebraic variety in $\mathfrak{g}_1$ defined over $\mathbb{R}$. We have $V_{\leq r-1} = \{x \in \mathfrak{g}_1 : \det P(x) = 0\}$. Since $\det P(x)$ is a power of the generic norm $\nu$ of the Jordan algebra $\mathfrak{g}_1$, the defining ideal $I(V_{\leq r-1})$ of $V_{\leq r-1}$ is generated by the irreducible polynomial $\nu$. The variety $V_{\leq k}$ is a conic variety, since $G_0^0$ contains the one-dimensional center acting on $\mathfrak{g}_1$ as homotheties. Therefore the defining ideal $I(V_{\leq k})$ of $V_{\leq k}$ is a homogeneous ideal. Let $I(V_{\leq k})_m$ denote the totality of homogeneous polynomials in $I(V_{\leq k})$ of degree $m$.

**Proposition 6.4.** *For a simple GLA (1.1) with $r \geq 2$, the singular locus $\mathrm{Sing}(V_{\leq 1})$ of the generalized determinantal variety $V_{\leq 1}$ in $\mathfrak{g}_1$ coincides with $V_0 = (0)$.*

*Proof.* Let $\mathfrak{a}_1$ be the linear span of $E_1, \cdots, E_r$ in $\mathfrak{g}_1$. Then it is known [1,8] that

$$\mathfrak{g}_1 = G_0^0 \mathfrak{a}_1. \tag{6.4}$$

First we claim that $I(V_{\leq 1})_1 = \emptyset$. Suppose the contrary. One can then choose a nonzero linear form $f$ on $\mathfrak{g}_1$ such that $f(V_1) = 0$. Since $r \geq 2$, there exists a point $x_0 \in \mathfrak{g}_1$ such that $f(x_0) \neq 0$. By (6.4) one can assume that $x_0$ lies in $\mathfrak{a}_1$. Since $E_i$ is conjugate to $E_1$ under $G_0^0$, $E_1, \cdots, E_r$ belong to $V_1$. By the assumption for $f$ we have that $f(E_i) = 0, 1 \leq i \leq r$, which implies that $f$ is identically zero on $\mathfrak{a}_1$. This contradicts the fact that $f(x_0) \neq 0$, which shows that the claim $I(V_{\leq 1})_1 = \emptyset$. Recall that the variety $V_{\leq 1}$ is defined over $\mathbb{R}$ (cf.6.2). One can choose a generator $\{f_1, \cdots, f_s\}$ of the ideal $I(V_{\leq 1})$ such that each polynomial $f_i$ is homogeneous and defined over $\mathbb{R}$. From the above argument, it follows that $\deg f_i \geq 2$. Consequently $(df_i)_0 = 0, 1 \leq i \leq s$, which shows that 0 is a singularity of $V_{\leq 1}$. Obviously we have that $\mathrm{Reg}(V_{\leq 1}) \supset V_1$. Therefore we conclude $\mathrm{Sing}(V_{\leq 1}) = V_0$. □

6.3. In this paragraph we treat the case where the GLA (1.1) is complex simple of $C_r$-type. The subspace $\mathfrak{g}_1$ is then a complex simple Jordan algebra. The following is a list of complex simple Jordan algebras.

| Type | $\mathfrak{g}_1$ | $r$ |
|---|---|---|
| I | $M_n(\mathbb{C})$ | $n$ |
| II | $\mathrm{Alt}_{2n}(\mathbb{C})$ | $n$ |
| III | $\mathrm{Sym}_n(\mathbb{C})$ | $n$ |
| IV | $\mathbb{C}^n$ | 2 |
| VI | $H_3(\mathbb{O}^{\mathbb{C}})$ | 3 |

Here $H_3(\mathbb{O}^{\mathbb{C}})$ denotes the exceptional simple Jordan algebra of $3 \times 3$ Hermitian matrices with entries in complex octonions $\mathbb{O}^{\mathbb{C}}$.

**Proposition 6.5.** *For any complex simple Jordan algebra $\mathfrak{g}_1$ with $r \geq 2$, we have*

$$\mathrm{Sing}(V_{\leq k}) = V_{\leq k-1}, \qquad 1 \leq k \leq r-1. \tag{6.5}$$

*Proof.*

(i) For the case of types I or III, $X \in V_{\leq k}$ if and only if $\mathrm{rk}\, X \leq k$. For the case of type II, $X \in V_{\leq k}$ if and only if $\mathrm{rk}\, X \leq 2k$. In those three cases, (6.5) is well-known (see for example [11]).

(ii) Consider the case of type IV. In this case we have $r = 2$ and $\mathfrak{g}_1 = V_2 \amalg V_1 \amalg V_0$. Therefore (6.5) follows from Proposition 6.4.

(iii) Now we consider the case of type VI. In this case $\mathfrak{g}_1$ can be identified with $H_3(\mathbb{O}^{\mathbb{C}})$ in a such a way that $E_i (i = 1, 2, 3)$ is sent to the diagonal matrix $(\delta_{i1}, \delta_{i2}, \delta_{i3})$. An element $x \in H_3(\mathbb{O}^{\mathbb{C}})$ is expressed as

$$x = \begin{pmatrix} \xi_1 & c & \bar{b} \\ \bar{c} & \xi_2 & a \\ b & \bar{a} & \xi_3 \end{pmatrix}, \qquad \xi_i \in \mathbb{C}, \quad a, b, c \in \mathbb{O}^{\mathbb{C}}. \tag{6.6}$$

The generic norm $\nu(x)$ of $x$ is given by

$$\nu(x) = \xi_1\xi_2\xi_3 - \xi_1 n(a) - \xi_2 n(b) - \xi_3 n(c) + t(abc),$$

where $n$ and $t$ denote respectively the norm and the trace of an octonion. We express an element $a \in \mathbb{O}^{\mathbb{C}}$ as $a = \sum_{i=0}^{7} a_i e_i$, where $\{e_i\}$ is the canonical basis of $\mathbb{O}^{\mathbb{C}}$. The variety $V_{\leq 2}$ is defined by the single equation $\nu(x) = 0, x \in \mathfrak{g}_1$. We then have

$$\begin{aligned} d\nu = &\left(\xi_2\xi_3 - n(a)\right) d\xi_1 + \left(\xi_1\xi_3 - n(b)\right) d\xi_2 + \left(\xi_1\xi_2 - n(c)\right) d\xi_3 \\ &+ \sum_{i=0}^{7}\left(-2\xi_1 a_i + \frac{\partial}{\partial a_i}t(abc)\right) da_i + \sum_{i=0}^{7}\left(-2\xi_2 b_i + \frac{\partial}{\partial b_i}t(abc)\right) db_i \\ &+ \sum_{i=0}^{7}\left(-2\xi_3 c_i + \frac{\partial}{\partial c_i}t(abc)\right) dc_i. \end{aligned} \tag{6.7}$$

Now let $x \in V_{\leq 1}$. From (6.4) and (6.3) it follows that there exists an element $g \in G_0^0$ such that $gx \in \mathfrak{a}_1 \cap V_{\leq 1}$, in other words, $gx$ is a diagonal matrix $\mathrm{diag}(\xi_1, \xi_2, \xi_3) \in V_{\leq 1}$, which implies that at least two of $\xi_1$, $\xi_2$, $\xi_3$ are zero. Therefore we have from (6.7) that

$$(d\nu)_{gx} = \xi_2\xi_3\, d\xi_1 + \xi_1\xi_3\, d\xi_2 + \xi_1\xi_2\, d\xi_3 = 0$$

Therefore, in view of the relative invariance of $\nu$ under $G_0$, we have $(d\nu)_x = 0$. By the Jacobian criterion, we obtain $V_{\leq 1} \subset \mathrm{Sing}(V_{\leq 2})$. On the other hand, clearly we have $V_2 \subset \mathrm{Reg}(V_{\leq 2})$. Consequently we conclude that $V_{\leq 1} = \mathrm{Sing}(V_{\leq 2})$. The equality $V_0 = \mathrm{Sing}(V_{\leq 1})$ follows from Proposition 6.4. □

We denote the defining ideal of an algebraic variety $A$ by $I(A)$.

**Corollary 6.6.** *Let $V_{\leq k}(1 \leq k \leq r-1)$ be a generalized determinantal variety in a complex simple Jordan algebra $\mathfrak{g}_1$. Then there exists a basis $\{f_1, \ldots, f_{s_k}\}$ of $I(V_{\leq k})$ such that each $f_i$ is defined over $\mathbb{R}$ and that $df_i \in I(\mathrm{Sing}(V_{\leq k})), 1 \leq i \leq s_k$, in other words, $(df_i)_p = 0$, $1 \leq i \leq s_k$ for each point $p \in \mathrm{Sing}(V_{\leq k})$.*

*Proof.*

Note that $V_{\leq k}$ is defined over $\mathbb{R}$ (cf.6.2). For types I and III, we choose, as a generator of $I(V_{\leq k})$, the totality of $(k+1)$-minors of a generic element of $\mathfrak{g}_1$. For Type II, we choose, as a generator of $I(V_{\leq k})$, the totality of the Pfaffians of principal $(2k+2)$-submatrices of a generic element of $\mathfrak{g}_1$. Then the assertion is well-known for those cases (cf.[11]). For the remaining two cases, the assertion was shown in the proof of Propositions 6.4 and 6.5. □

6.4. In this paragraph we wish to show that Proposition 6.5 is valid for a real simple Jordan algebra. Let us consider a real simple but not complex simple GLA(1.1) of $C_r$-type: $\mathfrak{g} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$. $r$ is the split rank of the symmetric pair $(\mathfrak{g}, \mathfrak{g}_0)$. In this case $\mathfrak{g}_1$ is a real simple but not complex simple Jordan algebra. Consider the complexification of the GLA $\mathfrak{g}$:

$$\mathfrak{g}^c = \mathfrak{g}_{-1}^c + \mathfrak{g}_0^c + \mathfrak{g}_1^c$$

Let $\bar{r}$ be the split rank of the symmetric pair $(\mathfrak{g}^c, \mathfrak{g}_0^c)$. The following is a list of real simple GLAs of $C_r$-type and their complexifications:

**Type I.**

$$\begin{cases} (\mathfrak{g}^c, \mathfrak{g}_0^c, \mathfrak{g}_1^c) = \left(\mathfrak{sl}(2n, \mathbb{C}), \mathfrak{sl}(n, \mathbb{C}) + \mathfrak{sl}(n, \mathbb{C}) + \mathbb{C}, M_n(\mathbb{C})\right), & \bar{r} = n, \\ (\mathfrak{g}, \mathfrak{g}_0, \mathfrak{g}_1) = \begin{cases} \left(\mathfrak{sl}(2n, \mathbb{R}), \mathfrak{sl}(n, \mathbb{R}) + \mathfrak{sl}(n, \mathbb{R}) + \mathbb{R}, M_n(\mathbb{R})\right), & r = n, \\ \left(\mathfrak{su}(n, n), \mathfrak{sl}(n, \mathbb{C}) + \mathbb{R}, H_n(\mathbb{C})\right), & r = n, \end{cases} \end{cases}$$

$$\begin{cases} (\mathfrak{g}^c, \mathfrak{g}_0^c, \mathfrak{g}_1^c) = \left(\mathfrak{sl}(4n, \mathbb{C}), \mathfrak{sl}(2n, \mathbb{C}) + \mathfrak{sl}(2n, \mathbb{C}) + \mathbb{C}, M_{2n}(\mathbb{C})\right), & \bar{r} = 2n, \\ (\mathfrak{g}, \mathfrak{g}_0, \mathfrak{g}_1) = \left(\mathfrak{sl}(2n, \mathbb{H}), \mathfrak{sl}(n, \mathbb{H}) + \mathfrak{sl}(n, \mathbb{H}) + \mathbb{R}, M_n(\mathbb{H})\right), & r = n, \end{cases}$$

**Type II.**

$$\begin{cases} (\mathfrak{g}^c, \mathfrak{g}_0^c, \mathfrak{g}_1^c) = \big(\mathfrak{so}(4n, \mathbb{C}), \mathfrak{gl}(2n, \mathbb{C}), \mathrm{Alt}_{2n}(\mathbb{C})\big), & \bar{r} = n, \\ (\mathfrak{g}, \mathfrak{g}_0, \mathfrak{g}_1) = \begin{cases} \big(\mathfrak{so}(2n, 2n), \mathfrak{gl}(2n, \mathbb{R}), \mathrm{Alt}_{2n}(\mathbb{R})\big), & r = n, \\ \big(\mathfrak{so}^*(4n), \mathfrak{gl}(n, \mathbb{H}), H_n(\mathbb{H})\big), & r = n \end{cases} \end{cases}$$

**Type III.**

$$\begin{cases} (\mathfrak{g}^c, \mathfrak{g}_0^c, \mathfrak{g}_1^c) = \big(\mathfrak{sp}(n, \mathbb{C}), \mathfrak{gl}(n, \mathbb{C}), \mathrm{Sym}_n(\mathbb{C})\big), & \bar{r} = n, \\ (\mathfrak{g}, \mathfrak{g}_0, \mathfrak{g}_1) = \big(\mathfrak{sp}(n, \mathbb{R}), \mathfrak{gl}(n, \mathbb{R}), \mathrm{Sym}_n(\mathbb{R})\big), & r = n, \end{cases}$$

$$\begin{cases} (\mathfrak{g}^c, \mathfrak{g}_0^c, \mathfrak{g}_1^c) = \big(\mathfrak{sp}(2n, \mathbb{C}), \mathfrak{gl}(2n, \mathbb{C}), \mathrm{Sym}_{2n}(\mathbb{C})\big), & \bar{r} = 2n, \\ (\mathfrak{g}, \mathfrak{g}_0, \mathfrak{g}_1) = \big(\mathfrak{sp}(n, n), \mathfrak{gl}(n, \mathbb{H}), \mathrm{SH}_n(\mathbb{H})\big), & r = n, \end{cases}$$

**Type IV.**

$$\begin{cases} (\mathfrak{g}^c, \mathfrak{g}_0^c, \mathfrak{g}_1^c) = \big(\mathfrak{so}(n+2, \mathbb{C}), \mathfrak{so}(n, \mathbb{C}) + \mathbb{C}, \mathbb{C}^n\big), & \bar{r} = 2, \\ (\mathfrak{g}, \mathfrak{g}_0, \mathfrak{g}_1) = \big(\mathfrak{so}(p+1, q+1), \mathfrak{so}(p, q) + \mathbb{R}, \mathbb{R}^n\big), & r = \begin{cases} 1 & (p = 0), \\ 2 & (p \geq 1), \end{cases} \\ p \leq q, \ p + q = n, \end{cases}$$

**Type VI.**

$$\begin{cases} (\mathfrak{g}^c, \mathfrak{g}_0^c, \mathfrak{g}_1^c) = \big(E_7^{\mathbb{C}}, E_6^{\mathbb{C}} + \mathbb{C}, H_3(\mathbb{O}^{\mathbb{C}})\big), & \bar{r} = 3, \\ (\mathfrak{g}, \mathfrak{g}_0, \mathfrak{g}_1) = \begin{cases} \big(E_{7(7)}, E_{6(6)} + \mathbb{R}, H_3(\mathbb{O}')\big), & r = 3, \\ \big(E_{7(-25)}, E_{6(-26)} + \mathbb{R}, H_3(\mathbb{O})\big), & r = 3. \end{cases} \end{cases}$$

In the above list, $\mathbb{H}$ denotes the quaternion algebra. $\mathbb{O}$, (resp. $\mathbb{O}'$) denotes the octonion (resp. split octonion) algebra. $H_n(\mathbb{F}), \mathbb{F} = \mathbb{C}, \mathbb{H}, \mathbb{O}, \mathbb{O}'$, denotes the Jordan algebra of $n \times n$ $\mathbb{F}$-Hermitian matrices. $\mathrm{SH}_n(\mathbb{H})$ denotes the Jordan algebra of $n \times n$ skew-Hermitian quaternion matrices. Note that the generalized determinantal varieties in a complex simple Jordan algebra are defined over $\mathbb{R}$. From a result of Takeuchi[14] we have

**Proposition 6.7.** *Let $\mathfrak{g}_1$ and $\mathfrak{g}_1^c$ be as above, and let $\mathfrak{g}_1 = \amalg_{k=0}^{r} V_k$ and $\mathfrak{g}_1^c = \amalg_{k=0}^{\bar{r}} \tilde{V}_k$ be the rank decomposition (cf.(6.2)) of $\mathfrak{g}_1$ and $\mathfrak{g}_1^c$, respectively. Suppose that $\bar{r} = r$. Then $V_{\leq k} (1 \leq k \leq r-1)$ coincides with the set of $\mathbb{R}$-rational points of the complex algebraic variety $\tilde{V}_{\leq k}$. Suppose that $\bar{r} = 2r$. Then $V_{\leq k} (1 \leq k \leq r-1)$ coincides with the sets of $\mathbb{R}$-rational points of the algebraic varieties $\tilde{V}_{\leq 2k}$ and of $\tilde{V}_{\leq 2k+1}$.*

From Proposition 6.7, we obtain

**Lemma 6.8.** *Let $f_1, \ldots, f_{s_k}$ be real polynomials on $\mathfrak{g}_1$, and let $\tilde{f}_i (1 \leq i \leq s_k)$ be the natural extension of $f_i$ to $\mathfrak{g}_1^c$. Then $I(V_{\leq k}) (1 \leq k \leq r-1)$ is generated by $f_1, \ldots, f_{s_k}$ if and only if $I(\tilde{V}_{\leq k})$ (resp. $I(\tilde{V}_{\leq 2k})$) is generated by $\tilde{f}_1, \ldots, \tilde{f}_{s_k}$ for $\bar{r} = r$ (resp. $\bar{r} = 2r$.).*

**Proposition 6.9.** *For a real simple (not complex simple) Jordan algebra $\mathfrak{g}_1$, we have*

$$\mathrm{Sing}(V_{\leq k}) = V_{\leq k-1}, \qquad 1 \leq k \leq r-1.$$

*Proof.* Let $\theta$ be the conjugation of $\mathfrak{g}_1^c$ with respect to $\mathfrak{g}_1$. Since $\tilde{V}_{\leq k}$ is $\theta$-stable, $\mathrm{Sing}(\tilde{V}_{\leq k})$ is also $\theta$-stable. Let $\big(\mathrm{Sing}(\tilde{V}_{\leq k})\big)_\theta$ be the set of $\theta$-fixed points in $\mathrm{Sing}(\tilde{V}_{\leq k})$. Suppose first $\bar{r} = r$. Since $\tilde{V}_{\leq k}$ is a conic variety defined over $\mathbb{R}$ (cf.6.2), one can choose a generator $\{\tilde{f}_1, \ldots, \tilde{f}_{s_k}\}$ of $I(\tilde{V}_{\leq k})$ such that each $\tilde{f}_i$ is homogeneous and defined over $\mathbb{R}$. By Corollary 6.6, $d\tilde{f}_i \in I(\mathrm{Sing}(\tilde{V}_{\leq k}))$. Let $f_i = \tilde{f}_i|_{\mathfrak{g}_1}$. Then $\{f_1, \ldots, f_{s_k}\}$ is a generator of $I(V_{\leq k})$, by Lemma 6.8. Let $p \in \big(\mathrm{Sing}(\tilde{V}_{\leq k})\big)_\theta$. Then $p \in \big(\tilde{V}_{\leq k}\big)_\theta = V_{\leq k}$, by Proposition 6.7. We have $(df_i)_p = (d\tilde{f}_i)_p = 0$, which implies that $p \in \mathrm{Sing}(\tilde{V}_{\leq k})$. Hence, by Propositions 6.5 and 6.7, we have

$$V_{\leq k-1} = (\tilde{V}_{\leq k-1})_\theta = \big(\mathrm{Sing}(\tilde{V}_{\leq k})\big)_\theta \subset \mathrm{Sing}(V_{\leq k}). \tag{6.8}$$

In view of the inclusion $V_k \subset \mathrm{Reg}(V_{\leq k})$, we conclude $V_{\leq k-1} = \mathrm{Sing}(\tilde{V}_{\leq k})_\theta = \mathrm{Sing}(V_{\leq k})$. As for the case $\bar{r} = 2r$, we should replace (6.8) by the equality

$$V_{\leq k-1} = (\tilde{V}_{\leq 2k-2})_\theta = (\tilde{V}_{\leq 2k-1})_\theta = \big(\mathrm{Sing}(\tilde{V}_{\leq 2k})\big)_\theta \subset \mathrm{Sing}(V_{\leq k}).$$

□

6.5. Combining Propositions 6.5 and 6.9, we have

**Theorem 6.10.** *Let $\mathfrak{g} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$ be a simple GLA of $C_r$-type, and let $\mathfrak{g}_1 = \amalg_{k=0}^r V_k$ be the rank decomposition. Then the closure $\bar{V}_k$ of $V_k$ is the generalized determinantal variety $V_{\leq k}$, and $\mathrm{Sing}(V_{\leq k}) = V_{\leq k-1}$ for $1 \leq k \leq r-1$.*

From Theorem 6.10 and Proposition 6.3 we have

**Theorem 6.11.** *Let $\mathfrak{g} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$ be a simple GLA, and let $r$ be the split rank of the symmetric pair $(\mathfrak{g}, \mathfrak{g}_0)$. Let $\mathfrak{g}_1 \oplus \mathfrak{g}_1' = \amalg_{k=0}^r M_k^*$ be the rank decomposition (cf. 5.4) of the Jordan pair $\mathfrak{g}_1 \oplus \mathfrak{g}_1'$. Then the closure $\bar{M}_k^*$ of $M_k^*$ in $\mathfrak{g}_1 \oplus \mathfrak{g}_1'$ is the algebraic variety $M_{\leq k}^*$, and*

$$\mathrm{Sing}(M_{\leq k}^*) = M_{\leq k-1}^*, \qquad \mathrm{Reg}(M_{\leq k}^*) = M_k^*, \qquad 1 \leq k \leq r-1.$$

6.6. Now we go back to the full $G$-orbits $M_k$.

**Lemma 6.12.** *$M_{\leq k}$ $(0 \leq k \leq r-1)$ is a real analytic set in $\tilde{M}$.*

*Proof.* Choose a point $p_0 \in M_{\leq k}$. Then one can find an element $g \in G$ such that $g(p_0) \in \mathfrak{g}_1 \oplus \mathfrak{g}_1'$. Choose a neighborhood $U$ of $p_0$ in $\tilde{M}$ in such a way that $U' := g(U) \subset \mathfrak{g}_1 \oplus \mathfrak{g}_1'$. Let $p \in U$. Then we have that $p \in U \cap M_{\leq k}$ if and only if $g(p) \in U' \cap M_{\leq k}^*$. Let $\{f_1, \ldots, f_{s_k}\}$ be a basis of the ideal $I(M_{\leq k}^*)$. Then $M_{\leq k}$ is expressed in $U$ as

$$U \cap M_{\leq k} = \{\, p \in U : (f_i \circ g)(p) = 0,\ 1 \leq i \leq s_k \,\},$$

which implies that $M_{\leq k}$ is a real analytic set of $\tilde{M}$. □

A point $p \in M_{\leq k}$ is a regular point of $M_{\leq k}$, if there exists a neighborhood $U$ of $p$ in $\tilde{M}$ such that $U \cap M_{\leq k}$ is a smooth manifold of dimension $d_k := \dim M_k$. Otherwise we say that $p$ is a singular point of $M_{\leq k}$. We denote by $\mathrm{Reg}(M_{\leq k})$ (resp. $\mathrm{Sing}(M_{\leq k})$) the regular (resp. singular) locus of $M_{\leq k}$. Finally we get the following theorem which gives the stratification of $\tilde{M}$ by $G$-orbits.

**Theorem 6.13.** *For $1 \leq k \leq r-1$, we have $\mathrm{Reg}(M_{\leq k}) = M_k$ and $\mathrm{Sing}(M_{\leq k}) = M_{\leq k-1}$.*

*Proof.* Let $p \in \mathrm{Reg}(M_{\leq k})$. Choose an element $g \in G$ such that $g(p) \in M_{\leq k}^*$. Then, since $p$ is a regular point of $M_{\leq k}$, $g(p)$ lies in $\mathrm{Reg}(M_{\leq k}^*) = M_k^*$ by Theorem 6.11. This implies $p \in M_k$, or equivalently, $\mathrm{Reg}(M_{\leq k}) = M_k$. Similarly we can show $\mathrm{Sing}(M_{\leq k}) = M_{\leq k-1}$. □

**Corollary 6.14.** *Suppose that a diffeomorphism $f$ of $\tilde{M}$ leaves the open orbit $M_r$ stable. Then $f$ leaves all other orbits $M_k$ $(0 \le k \le r-1)$ stable.*

*Proof.* By the assumption, $f$ leaves $M_{\le r-1}$ stable, and hence $\mathrm{Reg}(M_{\le r-1})$ is stable under $f$, that is, $f$ leaves $M_{r-1}$ stable, by Theorem 6.13. Therefore it leaves $M_{\le r-2}$ stable. Repeating this procedure, one concludes that all $G$-orbits $M_k$ are stable under $f$. □

## 7. Double foliation on the minimal boundary orbits

7.1. In this section, we always assume that $M$ is of $BC_r$-type.

In §2, we considered the double foliation on $\tilde{M}$, $\mathcal{M}^{\pm} = \{\, M^{\pm}(g_1 0^-, g_2 0^+) : g_1, g_2 \in G \,\}$. $\mathcal{M}^{\pm}$ naturally induce a double foliation $F_0^{\pm}$ on the minimal boundary orbit $M_0$. The leaves $F_0^{\pm}(p)$ of $F_0^{\pm}$ through a point $p \in M_0$ are given by the intersection $M^{\mp}(p) \cap M_0$.

**Lemma 7.1.** *The leaves of $F_0^{\pm}$ through the origin $(0^-, a_r 0^+) \in M_0$ are given by*

$$F_0^-(0^-, a_r 0^+) = U^-(0^-, a_r 0^+) = U^-/Q_r,$$
$$F_0^+(0^-, a_r 0^+) = a_r U^+ a_r^{-1}(0^-, a_r 0^+) = a_r U^+ a_r^{-1}/Q_r.$$

*Proof.* By the definition, $F_0^{\pm}(0^-, a_r 0^+) = M^{\mp}(0^-, a_r 0^+) \cap G(0^-, a_r 0^+)$. Let $(g0^-, g a_r 0^+) \in F_0^+(0^-, a_r 0^+)$, $g \in G$. Then $(g0^-, g a_r 0^+) \in M^-(0^-, a_r 0^+)$, which implies that $g a_r 0^+ = a_r 0^+$, or equivalently, $g \in a_r U^+ a_r^{-1}$. Conversely, let $u \in U^+$. Then $a_r u a_r^{-1}(0^-, a_r 0^+) = (a_r u a_r^{-1} 0^-, a_r 0^+) \in G(0^-, a_r 0^+) \cap M^-(0^-, a_r 0^+)$. □

**Lemma 7.2.** *The double foliation $F_0^{\pm}$ arises from the subspaces $\mathfrak{g}_1^{\pm}(r)$ of the GLA* (4.10).

*Proof.* Let $\mathfrak{u}^{\pm} = \mathrm{Lie}\, U^{\pm}$. By Lemma 7.1, the tangent spaces at $(0^-, a_r 0^+)$ to the leaves $F_0^{\pm}(0^-, a_r 0^+)$ are identified with the factor spaces $\mathfrak{u}^-/\mathfrak{q}_r$ and $(\mathrm{Ad}\, a_r)\mathfrak{u}^+/\mathfrak{q}_r$. By (4.15) and Lemma 4.4, we have

$$\mathfrak{u}^- = \mathfrak{g}_{-1} + \mathfrak{g}_0 = \mathfrak{g}_{-2}(r) + \mathfrak{g}_{-1}(r) + \mathfrak{g}_0(r) + \mathfrak{g}_1^+(r) = \mathfrak{q}_r + \mathfrak{g}_1^+(r). \tag{7.1}$$

Also, by (4.15) and Lemma 5.2, we have

$$\begin{aligned}(\mathrm{Ad}\, a_r)\mathfrak{u}^+ &= (\mathrm{Ad}\, a_r)\big(\mathfrak{g}_{-1}^-(r) + \mathfrak{g}_0(r) + \mathfrak{g}_1^+(r) + \mathfrak{g}_1^-(r) + \mathfrak{g}_2(r)\big) \\ &= \mathfrak{g}_1^-(r) + \mathfrak{g}_0(r) + \mathfrak{g}_{-1}^+(r) + \mathfrak{g}_{-1}^-(r) + \mathfrak{g}_{-2}(r) \\ &= \mathfrak{q}_r + \mathfrak{g}_1^-(r).\end{aligned} \tag{7.2}$$

Therefore $\mathfrak{u}^-/\mathfrak{q}_r$ and $(\mathrm{Ad}\, a_r)\mathfrak{u}^+/\mathfrak{q}_r$ can be identified with $\mathfrak{g}_1^+(r)$ and $\mathfrak{g}_1^-(r)$, respectively. □

Let $E := Z_r - 2Z$. Then $E$ is a central element of $\mathfrak{g}_0(r)$. It follows from (4.14) and (4.15) that

$$\mathrm{ad}\, E = \begin{cases} 0 & \text{on } \mathfrak{g}_{\mathrm{ev}}(r), \\ 1 & \text{on } \mathfrak{g}_{\pm 1}^+(r), \\ -1 & \text{on } \mathfrak{g}_{\pm 1}^-(r). \end{cases} \tag{7.3}$$

**Lemma 7.3.** *Let $g \in C(Z_r)$ and let $I = \mathrm{ad}_{\mathfrak{g}_1(r)} E$. Then the following three conditions are equivalent*:

(i) $g\big(\mathfrak{g}_1^{\pm}(r)\big) = \mathfrak{g}_1^{\pm}(r)$,
(ii) $gI = Ig$ *on* $\mathfrak{g}_1(r)$,
(iii) $g(E) = E$.

*Proof.* The only non-trivial assertion is the implication (ii) $\to$ (iii). Suppose (ii). Since $Z$, $Z_r \in \mathfrak{a}$, we have $\tau(Z_r) = -Z_r$ and $\tau(Z) = -Z$, and hence $\tau(E) = -E$. This implies that

$$\tau\big(\mathrm{ad}_{\mathfrak{g}_1(r)} E\big)\tau = -\mathrm{ad}_{\mathfrak{g}_{-1}(r)} E. \tag{7.4}$$

Consider the inner product $< \quad , \quad >$ on $\mathfrak{g}_1(r)$ defined by $< X, Y >= -(X, \tau Y)$. Let us denote by $g_\pm$ the restrictions of the actions of $g$ to $\mathfrak{g}_{\pm 1}(r)$, and denote by $g_+^*$ the adjoint operator of $g_+$ with respect to $< \quad , \quad >$. Then we have that $I$ is self-adjoint with respect to $< \quad , \quad >$, and hence, by (ii) we have $(g_+^*)^{-1} I = I (g_+^*)^{-1}$. We also have $g_- = \tau (g_+^*)^{-1} \tau$. Therefore it follows from (7.4) that

$$\begin{aligned} g_-(\mathrm{ad}_{\mathfrak{g}_{-1}(r)} E)(g_-)^{-1} &= \tau(g_+^*)^{-1}\tau(\mathrm{ad}_{\mathfrak{g}_{-1}(r)} E)\tau(g_+^*)\tau \\ &= -\tau(g_+^*)^{-1}(\mathrm{ad}_{\mathfrak{g}_1(r)} E)(g_+^*)\tau \\ &= -\tau I \tau \\ &= \mathrm{ad}_{\mathfrak{g}_{-1}(r)} E, \end{aligned}$$

which implies that $g_-$ commutes with $\mathrm{ad}_{\mathfrak{g}_{-1}(r)} E$. Combining this with (7.3) and (ii), we have that $g$ commutes with $\mathrm{ad}\, E$ on the whole $\mathfrak{g}$. This implies (iii). □

**Lemma 7.4.** $C(Z, Z_r) = \big\{\, g \in C(Z_r) : g\big(\mathfrak{g}_1^{\pm}(r)\big) = \mathfrak{g}_1^{\pm}(r) \,\big\}$.

*Proof.* Let $g \in C(Z_r)$. Then $g \in C(Z, Z_r)$ if and only if $g(E) = E$. Hence the assertion follows from Lemma 7.3. □

**Remark 7.1.** Lemmas 7.2, 7.4, Proposition 4.6 and Corollary 4.7 imply that our flag manifold $(M_0 = G/Q_r,\ F_0^{\pm})$ with double foliation $F_0^{\pm}$ is a so-called pseudo-product manifold associated to the simple GLA (4.10) with decomposition (4.14), in the sense of Tanaka [15].

7.2. We give a list of simple parahermitian symmetric spaces $M$ of $BC_r$-type and the corresponding minimal boundary orbits $M_0$. The list is obtained by extracting those ones satisfying (4.14) among all simple GLAs of the second kind classified in [7].

**Type I.** $(r = p)$.

$$M = \mathrm{SL}(n, \mathbb{F}) / \mathrm{S}\big(\mathrm{GL}(p, \mathbb{F}) \times \mathrm{GL}(n-p, \mathbb{F})\big), \qquad \mathbb{F} = \mathbb{R}, \mathbb{H}, \mathbb{C},$$
$$1 \le p < n - p,$$
$$M_0 = \begin{cases} \mathrm{SO}(n) / \mathrm{S}\big(\mathrm{O}(p) \times \mathrm{O}(n-2p) \times \mathrm{O}(p)\big), & \mathbb{F} = \mathbb{R}, \\ \mathrm{Sp}(n) / \mathrm{Sp}(p) \times \mathrm{Sp}(n-2p) \times \mathrm{Sp}(p), & \mathbb{F} = \mathbb{H}, \\ \mathrm{SU}(n) / \mathrm{S}\big(\mathrm{U}(p) \times \mathrm{U}(n-2p) \times \mathrm{U}(p)\big), & \mathbb{F} = \mathbb{C}. \end{cases}$$

**Type II.** $(r = n)$.

$$\begin{cases} M = \mathrm{SO}^0(2n+1, 2n+1) / \mathrm{GL}^0(2n+1, \mathbb{R}), \\ M_0 = \mathrm{SO}(2n+1) \times \mathrm{SO}(2n+1) / \mathrm{SO}(2n), \end{cases}$$
$$\begin{cases} M = \mathrm{SO}(4n+2, \mathbb{C}) / \mathrm{GL}(2n+1, \mathbb{C}), \\ M_0 = \mathrm{SO}(4n+2) / \mathrm{U}(2n) \cdot \mathbb{T}^1. \end{cases}$$

**Type V.**

$$\begin{cases} M = E_{6(6)}/\operatorname{Spin}(5,5)\cdot\mathbb{R}^+, & (r=2),\\ M_0 = \operatorname{Sp}(4)/\operatorname{Spin}(4)\times\operatorname{Spin}(4), \end{cases}$$

$$\begin{cases} M = E_{6(-26)}/\operatorname{Spin}(1,9)\cdot\mathbb{R}^+, & (r=1),\\ M_0 = F_4/\operatorname{Spin}(8), \end{cases}$$

$$\begin{cases} M = E_6^{\mathbb{C}}/\operatorname{Spin}(10,\mathbb{C})\cdot\mathbb{C}^*, & (r=2),\\ M_0 = E_6/\operatorname{Spin}(8)\cdot\mathbb{T}^2. \end{cases}$$

## 8. Determination of the automorphism groups of $M$

8.1. Let $(M = G/G_0, F^{\pm})$ be the parahermitian symmetric space associated with a simple GLA (1.1). In this paragraph we assume $M$ to be of $BC_r$-type. For the minimal boundary orbit $(M_0 = G/Q_r, F_0^{\pm})$ with double foliation $F_0^{\pm}$, we define the automorphism group by

$$\operatorname{Aut}(M_0, F_0^{\pm}) = \{\, g \in \operatorname{Diffeo}(M_0) : g_* F_0^{\pm} = F_0^{\pm} \,\}.$$

Tanaka [15] determined this group by establishing a Cartan connection on $M_0$ and by showing that $(M_0 = G/Q_r, F_0^{\pm})$ is the model space for the Cartan connection. Therefore, taking Remark 7.1 into account, we have

$$\operatorname{Aut}(M_0, F_0^{\pm}) = G. \tag{8.1}$$

**Theorem 8.1.** *Let $(M = G/G_0, F^{\pm})$ be a parahermitian symmetric space of $BC_r$-type associated with a simple GLA (1.1). Then* $\operatorname{Aut}(M, F^{\pm}) = \operatorname{Aut}(M_0, F_0^{\pm}) = G$.

*Proof.* We identify $M$ with its $\varphi$-image in $\tilde{M}$. Since $G$ acts on $M$ effectively and $F^{\pm}$ are $G$-invariant, the inclusion $G \subset \operatorname{Aut}(M, F^{\pm})$ is clear. Now let $f \in \operatorname{Aut}(M, F^{\pm})$. Then, by Lemma 2.4, $f$ preserves the fibers of the double fibration $M^- \xleftarrow{\pi^-} M \xrightarrow{\pi^+} M^+$. Hence $f$ induces the diffeomorphisms $f^{\pm}$ of $M^{\pm}$ such that $\pi^{\pm} \circ f = f^{\pm} \circ \pi^{\pm}$. Let $\tilde{f} := f^- \times f^+$. Clearly, the diffeomorphism $\tilde{f}$ preserves the product structure of $\tilde{M}$. We claim that $\tilde{f}|_M = f$. In fact, let $p \in M$, and let $q = f(p) \in M$. We write $p = (p^-, p^+)$ and $q = (q^-, q^+)$, where $p^{\pm}$, $q^{\pm} \in M^{\pm}$. The relation $\varpi^{\pm} \cdot \varphi = \pi^{\pm}$ (cf. §2) implies that $q^{\pm} = \varpi^{\pm}(q) = \varpi^{\pm}\big(f(p)\big) = f^{\pm}\big(\pi^{\pm}(p)\big) = f^{\pm}(p^{\pm})$. Hence $f(p) = (q^-, q^+) = \big(f^-(p^-), f^+(p^+)\big) = (f^- \times f^+)(p^-, p^+) = \tilde{f}(p)$. Since $\tilde{f}$ leaves $M$ invariant, by Corollary 6.14 $\tilde{f}$ leaves $M_0$ invariant. Obviously $f_0 := \tilde{f}|_{M_0}$ belongs to $\operatorname{Aut}(M_0, F_0^{\pm})$. We wish to show that $\tilde{f}$ can be uniquely recovered by its restriction $f_0$. Corresponding to the expression $\tilde{M} = M^- \times (M^+)_r$, one can express $\tilde{f}$ as $\tilde{f} = f_1 \times f_2$, where $f_1$ and $f_2$ are diffeomorphisms of $M^-$ and $(M^+)_r$, respectively. It follows from Lemma 7.1 that the leaves of its double foliation $F_0^{\pm}$ arise as the fibers of the double fibration $M^- \xleftarrow{\pi_0^-} M_0 \xrightarrow{\pi_0^+} (M^+)_r$ given in Theorem 3.1 (iv). Moreover this double fibration of $M_0$ is just the restriction of the trivial double fibration $M^- \xleftarrow{\varpi_0^-} \tilde{M} \xrightarrow{\varpi_0^+} (M^+)_r$ (cf. (5.3)). Therefore, if we denote by $f_0^-$ and $f_0^+$ the diffeomorphisms of $M^-$ and $(M^+)_r$ induced by $f_0$, then it follows that $f_0^- = f_1$ and $f_0^+ = f_1$. We have thus shown that $\tilde{f}$ is uniquely recovered from $f_0$. As a result, the correspondence $f \mapsto f_0$ is an injective homomorphism of $\operatorname{Aut}(M, F^{\pm})$ into $\operatorname{Aut}(M_0, F_0^{\pm})$. Consequently, in view of (8.1) we have that $\operatorname{Aut}(M, F^{\pm}) = G$. □

8.2. In this paragraph we are concerned with $C_r$-type. Under this assumption, for the GLA (4.10) we have $\mathfrak{g}_{\pm 1}(r) = (0)$, $\mathfrak{g}_{\pm 2}(r) = \mathfrak{g}_{\pm 1}$, $\mathfrak{g}_0(r) = \mathfrak{g}_0$, $Z_r = Z$ and hence $C(Z_r) =$

$C(Z) = G_0$. The rank decomposition (5.12) becomes $\mathfrak{g}_1 = \coprod_{k=0}^{r} V_k$, where $V_k$ is a union of equi-dimensional $G_0$-orbits. Now consider the $G_0$-stable conic algebraic set $V_{\leq r-1}$, which is the boundary $\partial V_r$ of $V_r$. One can extend the cone $\partial V_r$ to a cone field on the whole $M^-$ by using the $G$-action on $M^-$. We call the cone field a generalized conformal structure $\mathcal{K}$ ([2]). One can consider the automorphism group $\mathrm{Aut}(M^-, \mathcal{K})$, the totality of diffeomorphisms leaving the cone field $\mathcal{K}$ invariant. This group was determined for each symmetric $R$-space $M^-$ ([2]):

$$\mathrm{Aut}(M^-, \mathcal{K}) = \begin{cases} G, & r \geq 2, \\ \mathrm{Diffeo}(M^-), & r = 1. \end{cases} \tag{8.2}$$

Recall that $U^- = a_r\, U^+ a_r^{-1}$ for $C_r$-type (cf. Theorem 3.1). This is equivalent to the condition $a_r\, 0^+ = 0^-$, and the new origin $(0^-, a_r\, 0^+)$ becomes $(0^-, 0^-)$. By (5.3), $\tilde{M}$ takes the form $\tilde{M} = M^-(0^-, 0^-) \times M^+(0^-, 0^-) = M^- \times M^-$. Further the minimal boundary orbit $M_0$ becomes $M_0 = G(0^-, 0^-) = G/U^- = M^-$, the diagonal set of $\tilde{M} = M^- \times M^-$.

Now let $f \in \mathrm{Aut}(M, F^{\pm})$, and let $\tilde{f} = f^- \times f^+$ be the extension of $f$ to $\tilde{M}$ given in 8.1. Let us express $\tilde{f}$ as $\tilde{f} = f_1 \times f_2$ corresponding to the expression $\tilde{M} = M^- \times M^-$. By Corollary 6.14, $\tilde{f}$ leaves $M_0$, the diagonal of $\tilde{M}$, invariant, from which we have $f_1 = f_2$, that is, $\tilde{f} = f_1 \times f_1$. Thus it follows that the correspondence $f \mapsto f_1$ is an injective homomorphism of $\mathrm{Aut}(M, F^{\pm})$ into $\mathrm{Diffeo}(M^-)$. The following lemma is essentially due to Tanaka [15].

**Lemma 8.2.** *Suppose that $(0^-, 0^-)$ is a fixed point of $\tilde{f}$. Then the differential $(f_1)_*$ at $0^-$ leaves the cone $\partial V_r$ stable.*

*Proof.* Let $Y \in \partial V_r$. Then $(0, tY)$ is a path in $M^*_{\leq r-1}$. By the assumption, the curve $\tilde{f}(0, tY)$ lies in $M^*_{\leq r-1}$ for $|t|$ sufficiently small. Therefore $\Phi\big(\tilde{f}(0, tY)\big) = \Phi\big(f_1(0), f_1(tY)\big) = f_1(tY) - f_1(0) = f_1(tY)$ lies in $\partial V_r$. Hence $(f_1)_{*0^-}(Y) = \lim_{t \to 0} \frac{1}{t} f_1(tY) \in \partial V_r$. □

**Lemma 8.3.** *Let $\tilde{f} = f_1 \times f_1$ be the extension of $f \in \mathrm{Aut}(M, F^{\pm})$ to $\tilde{M} = M^- \times M^-$. Then $f_1 \in \mathrm{Aut}(M^-, \mathcal{K})$.*

*Proof.* Let $\mathcal{K} = \{(\partial V_r)_p\}_{p \in M^-}$, where $(\partial V_r)_p$ denotes the cone at a point $p \in M^-$ belonging to the field $\mathcal{K}$. Note that, if $p = b \cdot 0^-$, $b \in G$, then $(\partial V_r)_p$ is just the cone $b_*(\partial V_r)$. We have to show that $(f_1)_*(\partial V_r)_p = (\partial V_r)_{f_1(p)}$. Choose an element $a \in G$ such that $a^{-1} f_1 b(0^-) = 0^-$. Then the transformation $a^{-1} \tilde{f} b$ on $\tilde{M}$ is the extension of $a^{-1} f b \in \mathrm{Aut}(M, F^{\pm})$. Decompose $a^{-1} f b$ as $a^{-1} f_1 b \times a^{-1} f_1 b$ corresponding to the decomposition $\tilde{M} = M^- \times M^-$. By Lemma 8.3, we see that $(a^{-1} f_1 b)_{*0^-}$ leaves $\partial V_r$ invariant. Consequently we have

$$(f_1)_{*p}(\partial V_r)_p = (f_1)_{*p}(\partial V_r)_{b \cdot 0^-} = (f_1)_{*p} b_{*0^-}(\partial V_r) = a_{*0^-}(\partial V_r) = (\partial V_r)_{a \cdot 0^-} = (\partial V_r)_{f_1(p)}.$$

This implies that $f_1 \in \mathrm{Aut}(M^-, \mathcal{K})$. □

**Theorem 8.4.** *Let $(M = G/G_0, F^{\pm})$ be the parahermitian symmetric space of $C_r$-type associated with a simple GLA (1.1), and let $\mathcal{K}$ be the above generalized conformal structure on the symmetric $R$-space $M^- = G/U^-$. Then*

$$\mathrm{Aut}(M, F^{\pm}) = \mathrm{Aut}(M^-, \mathcal{K}) = \begin{cases} G, & r \geq 2, \\ \mathrm{Diffeo}(M^-), & r = 1. \end{cases}$$

*Proof.* Suppose $r \geq 2$. As we noted before Lemma 8.2, the correspondence $\mathrm{Aut}(M, F^{\pm}) \ni f \mapsto f_1 \in \mathrm{Diffeo}(M^-)$ is injective. But, by Lemma 8.3, the image $f_1$ lies in $\mathrm{Aut}(M^-, \mathcal{K})$. Hence we have the injective homomorphism $\mathrm{Aut}(M, F^{\pm}) \hookrightarrow \mathrm{Aut}(M^-, \mathcal{K})$. Since $G$ is a

subgroup of $\mathrm{Aut}(M, F^{\pm})$, it follows from (8.2) that $G \subset \mathrm{Aut}(M, F^{\pm}) \simeq \mathrm{Aut}(M^-, \mathcal{K}) = G$. Suppose next $r = 1$. Then the $G$-orbit decomposition of $\tilde{M}$ leaves $\tilde{M} = M \amalg M^-$. So, for any diffeomorphism $f_1$ of $M^-$, $(f_1 \times f_1)|_M$ is an element of $\mathrm{Aut}(M, F^{\pm})$. Therefore we have $\mathrm{Aut}(M, F^{\pm}) \simeq \mathrm{Aut}(M^-, \mathcal{K}) = \mathrm{Diffeo}(M^-)$ (cf. (8.27)). □

**Remark 8.1.** As is seen in the table in 6.4, the parahermitian symmetric space $M$ of $C_1$-type is $\mathrm{SO}^0(1, q+1)/\mathrm{SO}(q) \cdot \mathbb{R}^+$, and the corresponding symmetric $R$-space $M^-$ is the conformal $q$-sphere.

**Remark 8.2.** In case where $\mathrm{Aut}(M, F^{\pm}) = G$ in Theorems 8.1 and 8.4, we also have

$$\mathrm{Aut}(M, F^{\pm}) = \mathrm{Aut}(M, F^{\pm}, \omega).$$

**Remark 8.3.** A parahermitian symmetric space $M = G/H$ associated to a simple GLA (1.1) is diffeomorphic to the cotangent bundle of the associated symmetric $R$-space $M^- = G/U^-$. Let $M^-$ be the quaternionic Grassmannian $\mathrm{Gr}_2(\mathbb{H}^4)$ of quaternionic 2-planes in $\mathbb{H}^4$. There are two parahermitian symmetric spaces:

$$\mathrm{SL}(4, \mathbb{H})/\mathrm{SL}(2, \mathbb{H}) \times \mathrm{SL}(2, \mathbb{H}) \times \mathbb{R}^+ \quad \text{and} \quad E_{6(6)}/\mathrm{Spin}(5, 5) \cdot \mathbb{R}^+,$$

which have $\mathrm{Gr}_2(\mathbb{H}^4)$ as the associated symmetric $R$-spaces. The first one is of $C_2$-type; the second one is of $BC_2$-type. Theorems 8.1 and 8.4 tell us that the cotangent bundle of $\mathrm{Gr}_2(\mathbb{H}^4)$ has two different paracomplex structures which are both homogeneous.

## References


1. J. Faraut, S. Kaneyuki, A. Koranyi, Q.-K. Lu, and G. Roos, *Analysis and geometry on complex homogeneous domains*, Progress in Mathematics, no. 185, Birkhäuser, 2000.
2. S. Gindikin and S. Kaneyuki, *On the automorphism group of the generalized conformal structure of a symmetric R-space*, Differential Geometry and its Applications **8** (1998), 21–33.
3. A. Gyoja and H. Yamashita, *Associated variety, Kostant-Sekiguchi correspondence, and locally free $U(n)$-action on Harish-Chandra modules*, Journal of the Mathematical Society of Japan **5** (1999), 129–149.
4. Z. Hou, S. Deng, S. Kaneyuki, and K. Nishiyama, *Dipolarizations in semisimple Lie algebras and homogeneous parakähler manifolds*, Journal of Lie Theory **9** (1999), 215–232.
5. S. Kaneyuki, *On classification of parahermitian symmetric spaces*, Tokyo Journal of Mathematics **8** (1985), 473–482.
6. ———, *On orbit structure of compactifications of parahermitian symmetric spaces*, Japan. Journal of Mathematics **13** (1987), 333–370.
7. ———, *On the subalgebras $\mathfrak{g}_0$ and $\mathfrak{g}_{\mathrm{ev}}$ of semisimple graded Lie algebras*, Journal of the Mathematical Society of Japan **45** (1993), 1–19.
8. ———, *The Sylvester's law of inertia in simple graded Lie algebras*, Journal of the Mathematical Society of Japan **50** (1998), 593–614.
9. ———, *Compactification of parahermitian symmetric spaces and its applications, I: Tube type realizations*, Proceedings of the III International Workshop, Lie Theory and Its Applications in Physics (H. D. Doebner, V. K. Dobrev, and J. Hilgert, eds.), World Scientific Publishers, 2000, pp. 63–74.
10. S. Kaneyuki and M. Kozai, *Paracomplex structures and affine symmetric spaces*, Tokyo Journal of Mathematics **8** (1985), 81–98.
11. T. Levasseur and J. T. Stafford, *Rings of differential operators on classical rings of invariants*, Memoirs of the American Mathematical Society, no. 412, 1989.
12. G. Ólafsson and B. Ørsted, *Causal compactification and Hardy spaces*, Transactions of the American Mathematical Society **351** (1999), 3771–3792.
13. M. Takeuchi, *On conjugate loci and cut loci of compact symmetric spaces II*, Tsukuba Journal of Mathematics **3** (1979), 1–29.

14. _____, *Basic transformations of symmetric R-spaces*, Osaka Journal of Mathematics **25** (1988), 259–297.
15. N. Tanaka, *On affine symmetric spaces and the automorphism groups of product manifolds*, Hokkaido Mathematics Journal **14** (1985), 277–351.

Department of Mathematics, Nihon Institute of Technology, Miyashiro-cho, Saitama 345-8501, Japan

*E-mail address*: kaneyuki@hoffman.cc.sophia.ac.jp