BIMODULE DEFORMATIONS, PICARD GROUPS AND CONTRAVARIANT CONNECTIONS

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Abstract. We study deformations of invertible bimodules and the behavior of Picard groups under deformation quantization. While K_0-groups are known to be stable under formal deformations of algebras, Picard groups may change drastically. We identify the semiclassical limit of bimodule deformations as contravariant connections and study the associated deformation quantization problem. Our main focus is on formal deformation quantization of Poisson manifolds by star products.

1. Introduction

Some important mathematical formulations of quantization are based on replacing a classical algebra of observables $\mathcal{A}$ by a noncommutative one, obtained from $\mathcal{A}$ through a deformation procedure [2] (see also [24, 28] for more analytical approaches). It is natural to study the behavior of algebraic invariants, such as K-theory and Hochschild (co)homology, in this process, see e.g. [26, 27, 29, 32]. Building on [8], the main objective of this paper is to discuss the behavior of Picard groups under formal deformation quantization.

More precisely, let $\mathcal{A}$ be a commutative unital algebra over a commutative, unital ring k, and let $\boldsymbol{\mathcal{A}} = (\mathcal{A}[[\lambda]], \star)$ be a formal deformation of $\mathcal{A}$ [16]. The classical limit map

$$\mathrm{cl} : \boldsymbol{\mathcal{A}} \longrightarrow \mathcal{A}, \quad \sum_{r=0}^{\infty} \lambda^r a_r \mapsto a_0, \tag{1.1}$$

induces a group homomorphism

$$\mathrm{cl}_* : \mathrm{Pic}(\boldsymbol{\mathcal{A}}) \longrightarrow \mathrm{Pic}(\mathcal{A}), \tag{1.2}$$

and our goal is to study the kernel and image of cl_*, with special focus on the case where the deformations come from star products on Poisson manifolds. Recall that the Picard group of an arbitrary unital algebra is defined as the group of isomorphism classes of invertible bimodules over it (see Definition 2.1); for example, if $\mathcal{A} = C^\infty(M)$ is the algebra of complex-valued smooth functions on a manifold M, then $\mathrm{Pic}(\mathcal{A})$ is the semi-direct product

$$\mathrm{Pic}(C^\infty(M)) = \mathrm{Diff}(M) \ltimes \mathrm{Pic}(M),$$

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where $\mathrm{Diff}(M)$ is the diffeomorphism group and $\mathrm{Pic}(M) \cong H^2(M, \mathbb{Z})$ is the *geometric* Picard group, consisting of isomorphism classes of complex line bundles over M; the action is given by pull-back on line bundles.

The behavior of K_0-groups under deformations was discussed in [7, 29]: the classical limit map (1.1) induces a map of K_0-groups,

$$K_0(\boldsymbol{\mathcal{A}}) \longrightarrow K_0(\mathcal{A}),$$

which turns out to be a group isomorphism [29, Thm. 4]. In the case of (1.2), we will see that cl_* is far from being an isomorphism in general.

In the purely algebraic setting, the kernel of cl_* is in one-to-one correspondence with outer self-equivalences of $\boldsymbol{\mathcal{A}}$, i.e., the group of automorphisms of $\boldsymbol{\mathcal{A}}$ of the form $T = \mathrm{Id} + O(\lambda)$ modulo inner automorphisms; the image of cl_* is described in terms of a canonical action Φ of $\mathrm{Pic}(\mathcal{A})$ on the moduli space of equivalence classes of formal deformations of $\mathcal{A}$ [6]. Roughly speaking, two deformations $\star'$ and $\star$ of $\mathcal{A}$ are in the same Φ-orbit if and only if there exists an $(\mathcal{A}, \mathcal{A})$-invertible bimodule X that can be deformed into a $(\star', \star)$-equivalence bimodule. In fact, the Φ-orbits characterize Morita equivalent deformations of $\mathcal{A}$.

Just as associative algebraic deformations of $\mathcal{A}$ correspond to Poisson brackets, the semiclassical limit of a $(\star', \star)$-bimodule deformation of an $(\mathcal{A}, \mathcal{A})$-invertible bimodule X is an algebraic generalization of what is known in Poisson geometry as a linear *contravariant connection* [14, 20, 30]. We note that contravariant connections on X arising in the semiclassical limit of a $(\star', \star)$-bimodule deformation must have a fixed curvature τ, depending only on $\star'$ and $\star$. More generally, we discuss the existence and classification of bimodule deformations "in the direction" of a fixed contravariant connection, in the same spirit of deformation quantization of Poisson algebras.

In the geometric context of star products on Poisson manifolds, our results are as follows. We consider star products satisfying two extra conditions (see conditions C1) and C2) in Sections 5.3 and 6) relating the Poisson center and the Poisson derivations to the center and derivations of the star product. These conditions are always satisfied, e.g., by symplectic star products and, on arbitrary Poisson manifolds, by the star products constructed in [10, 11]. For star products on a Poisson manifold (M, π) satisfying these conditions, we show that, for a fixed line bundle with contravariant connection satisfying the curvature condition, there always exist corresponding bimodule deformations. Moreover, equivalence classes of bimodule deformations in the direction of a given contravariant connection are in bijection with $H^1_\pi(M, \mathbb{C})[[\lambda]]$, the set of formal power series with coefficients in the first Poisson cohomology group.

The kernel of cl_* is in bijection with

$$\frac{H^1_\pi(M, \mathbb{C})}{2\pi i H^1_\pi(M, \mathbb{Z})} + \lambda H^1_\pi(M, \mathbb{C})[[\lambda]]. \tag{1.3}$$

In particular, cl_* is injective if and only if $H^1_\pi(M, \mathbb{C}) = 0$.

Finally, we give a description of the image of cl_* for symplectic star products. As an example, we show that if $[\psi^*(\star)] = [\star]$ for all symplectomorphisms ψ (here $[\star]$ denotes the equivalence class of the star product $\star$), then we can write the image of

cl_* as the semi-direct product

$$\mathrm{cl}_*(\mathrm{Pic}(\boldsymbol{\mathcal{A}})) = \mathrm{Sympl}(M) \ltimes \mathrm{Tor}(H^2(M,\mathbb{Z})),$$

where $\mathrm{Sympl}(M)$ denotes the group of symplectomorphisms of M and $\mathrm{Tor}(H^2(M,\mathbb{Z}))$ is the subgroup of torsion elements in $H^2(M,\mathbb{Z})$; the action is given by pull-back. This is the case for arbitrary star products on Riemann surfaces and on $\mathbb{C}P^n$. We also construct explicit examples where the image of the map $\mathrm{cl}_*^r = \mathrm{pr} \circ \mathrm{cl}_*$,

$$\mathrm{Pic}(\boldsymbol{\mathcal{A}}) \xrightarrow{\mathrm{cl}_*} \mathrm{Pic}(\mathcal{A}) = \mathrm{Diff}(M) \ltimes H^2(M,\mathbb{Z}) \xrightarrow{\mathrm{pr}} H^2(M,\mathbb{Z})$$

contains non-torsion elements, but it seems hard to describe exactly how big this image can be. Nevertheless, we show that cl_*^r is onto if and only if $H^2(M,\mathbb{Z})$ only contains torsion elements.

The paper is organized as follows. In Section 2, we review the basics of equivalence bimodules and Picard groups. In Section 3, we define the map cl_* and give descriptions of its kernel and image in purely algebraic terms. Section 4 discusses bimodule deformations and the contravariant connections in their semiclassical limit. Section 5 collects results on the structure of the automorphism group of star product algebras, used in Section 6 to discuss deformation quantization of line bundles over Poisson manifolds. Finally, Section 7 describes the kernel and the image of cl_* in cohomological terms for the case of star products.

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2. Preliminaries on equivalence bimodules and Picard groups

In this section we recall definitions and standard facts about equivalence bimodules and Picard groups. The reader should consult [1] for details.

2.1. **Equivalence bimodules and Picard groups.** Let k be a commutative unital ring of characteristic zero, and let $\mathcal{A}$ and $\mathcal{B}$ be associative unital algebras over k. For convenience we shall assume $\mathbb{Q} \subseteq k$.

Definition 2.1. A $(\mathcal{B},\mathcal{A})$-bimodule X is called an *equivalence bimodule* if there exists an $(\mathcal{A},\mathcal{B})$-bimodule X^* so that $\mathrm{X} \otimes_{\mathcal{A}} \mathrm{X}^* \cong \mathcal{B}$ as $(\mathcal{B},\mathcal{B})$-bimodules and $\mathrm{X}^* \otimes_{\mathcal{B}} \mathrm{X} \cong \mathcal{A}$ as $(\mathcal{A},\mathcal{A})$-bimodules.

Let $\underline{\mathrm{EBim}}(\mathcal{B},\mathcal{A})$ denote the category of $(\mathcal{B},\mathcal{A})$-equivalence bimodules, with bimodule homomorphisms as morphisms. The set of isomorphism classes of objects in $\underline{\mathrm{EBim}}(\mathcal{B},\mathcal{A})$ is denoted by $\mathrm{EBim}(\mathcal{B},\mathcal{A})$. When $\mathcal{A} = \mathcal{B}$, we denote $\underline{\mathrm{EBim}}(\mathcal{A},\mathcal{A})$ by $\underline{\mathrm{Pic}}(\mathcal{A})$, and objects in $\underline{\mathrm{Pic}}(\mathcal{A})$ are also called $(\mathcal{A},\mathcal{A})$-*invertible bimodules.* In this case, the set $\mathrm{EBim}(\mathcal{A},\mathcal{A})$ forms a group under $\otimes_{\mathcal{A}}$, denoted by $\mathrm{Pic}(\mathcal{A})$.

Definition 2.2. The group $\mathrm{Pic}(\mathcal{A})$ is called the *Picard group* of $\mathcal{A}$.

We denote the isomorphism class of an object X in $\underline{\mathrm{EBim}}(\mathcal{B},\mathcal{A})$ by $[\mathrm{X}] \in \mathrm{EBim}(\mathcal{B},\mathcal{A})$. We recall a few facts from Morita theory [25], see also e.g. [23].

(1) Let ${}_{\mathcal{A}}\mathfrak{M}$ (resp. ${}_{\mathcal{B}}\mathfrak{M}$) denote the category of left $\mathcal{A}$-modules (resp. left $\mathcal{B}$-modules). Each object X in $\underline{\mathrm{EBim}}(\mathcal{B},\mathcal{A})$ defines a functor

$$\mathfrak{F}_{\mathrm{X}} = \mathrm{X}\otimes_{\mathcal{A}} \cdot : {}_{\mathcal{A}}\mathfrak{M} \longrightarrow {}_{\mathcal{B}}\mathfrak{M}, \quad \mathfrak{F}_{\mathrm{X}}(\mathrm{M}) = \mathrm{X}\otimes_{\mathcal{A}} \mathrm{M}, \tag{2.1}$$

which is an equivalence of categories. Conversely, any equivalence functor $\mathfrak{F} : {}_{\mathcal{A}}\mathfrak{M} \longrightarrow {}_{\mathcal{B}}\mathfrak{M}$ is isomorphic to a functor of the form (2.1), for some $\mathrm{X} \in \underline{\mathrm{EBim}}(\mathcal{B},\mathcal{A})$. When $\mathcal{A}=\mathcal{B}$, this correspondence induces an isomorphism between $\mathrm{Pic}(\mathcal{A})$ and the group of isomorphism classes of self-equivalence functors of ${}_{\mathcal{A}}\mathfrak{M}$, with group operation given by composition.

(2) If $\mathrm{X} \in \underline{\mathrm{EBim}}(\mathcal{B},\mathcal{A})$, then $[\mathrm{X}^*] = [\mathrm{Hom}_{\mathcal{A}}(\mathrm{X}_{\mathcal{A}},\mathcal{A})]$ and we have an algebra isomorphism $\mathrm{End}_{\mathcal{A}}(\mathrm{X}_{\mathcal{A}}) \cong \mathcal{B}$ via the $\mathcal{B}$-left action. In particular, this action is non-degenerate: $bx = 0$ for all $x \in \mathrm{X}$ implies that $b = 0$. Similar results hold for the right action of $\mathcal{A}$.

(3) Any $\mathrm{X} \in \underline{\mathrm{EBim}}(\mathcal{B},\mathcal{A})$ is finitely generated, projective and full both as a left $\mathcal{B}$-module and right $\mathcal{A}$-module. In particular, as a right $\mathcal{A}$-module, $\mathrm{X}_{\mathcal{A}} \cong P\mathcal{A}^n$, where $P \in M_n(\mathcal{A})$ is a full idempotent and $n \in \mathbb{N}$; fullness means that $M_n(\mathcal{A})PM_n(\mathcal{A}) = M_n(\mathcal{A})$. As a result of Morita's theorem [25], $\mathrm{X} \in \underline{\mathrm{EBim}}(\mathcal{B},\mathcal{A})$ if and only if there exists a full idempotent $P \in M_n(\mathcal{A})$ such that $\mathrm{X}_{\mathcal{A}} \cong P\mathcal{A}^n$ as right $\mathcal{A}$-modules and $PM_n(\mathcal{A})P \cong \mathcal{B}$. This shows in particular that $\mathrm{EBim}(\mathcal{B},\mathcal{A})$ and $\mathrm{Pic}(\mathcal{A})$ are indeed sets.

2.2. **The contribution of algebra automorphisms.** We denote the group of automorphisms of $\mathcal{A}$ by $\mathrm{Aut}(\mathcal{A})$, and $\mathrm{InnAut}(\mathcal{A})$ denotes the normal subgroup of inner automorphism, i.e., automorphisms of the form $\mathrm{Ad}(u) : a \mapsto uau^{-1}$ for an invertible element $u \in \mathcal{A}$.

Let $\mathrm{X} \in \underline{\mathrm{EBim}}(\mathcal{B},\mathcal{A})$, $\phi \in \mathrm{Aut}(\mathcal{B})$ and $\psi \in \mathrm{Aut}(\mathcal{A})$. We define a new element

$${}_{\phi}\mathrm{X}_{\psi} \in \underline{\mathrm{EBim}}(\mathcal{B},\mathcal{A})$$

by setting ${}_{\phi}\mathrm{X}_{\psi} \cong \mathrm{X}$ as an additive group and changing the left and right actions by

$$b\cdot x := \phi(b)x \qquad \text{and} \qquad x\cdot a := x\psi(a),$$

for $b \in \mathcal{B}$, $a \in \mathcal{A}$ and $x \in \mathrm{X}$. In this notation, $\mathrm{X} = {}_{\mathrm{Id}}\mathrm{X}_{\mathrm{Id}}$. It is easy to check that

$${}_{\phi}\mathrm{X}_{\psi} \cong {}_{\phi}\mathcal{B}\otimes_{\mathcal{B}} \mathrm{X}\otimes_{\mathcal{A}} \mathcal{A}_{\psi}. \tag{2.2}$$

Proposition 2.3. *Let $\mathrm{X},\mathrm{Y} \in \underline{\mathrm{EBim}}(\mathcal{B},\mathcal{A})$ be such that $\mathrm{X}_{\mathcal{A}} \cong \mathrm{Y}_{\mathcal{A}}$ (as right $\mathcal{A}$-modules). Then there exists $\phi \in \mathrm{Aut}(\mathcal{B})$ so that ${}_{\phi}\mathrm{X}_{\mathrm{Id}} \cong \mathrm{Y}$ (as $(\mathcal{B},\mathcal{A})$-bimodules). Moreover, ${}_{\phi}\mathrm{X}_{\mathrm{Id}} \cong \mathrm{X}$ as $(\mathcal{B},\mathcal{A})$-bimodules if and only if $\phi \in \mathrm{InnAut}(\mathcal{B})$.*

Proof. Let $j : \mathrm{X}_{\mathcal{A}} \longrightarrow \mathrm{Y}_{\mathcal{A}}$ be a right $\mathcal{A}$-module isomorphism. Fix $b \in \mathcal{B}$ and consider the map from X to itself defined by

$$x \quad \mapsto \quad j^{-1}(bj(x)).$$

Since this map is right $\mathcal{A}$-linear and $\mathcal{B} \cong \mathrm{End}_{\mathcal{A}}(\mathrm{X}_{\mathcal{A}})$, it defines a unique element $\tilde{b} \in \mathcal{B}$ so that $\tilde{b}\cdot x = j^{-1}(bj(x))$. This procedure defines a map

$$\phi : \mathcal{B} \longrightarrow \mathcal{B}, \;\; \phi(b) = \tilde{b},$$

which can be easily seen to be an automorphism of $\mathcal{B}$. From the equation $j(\phi(b)x) = bj(x)$, it follows that

$$j : {}_{\phi}\mathrm{X}_{\mathrm{Id}} \longrightarrow \mathrm{Y}$$

is a bimodule isomorphism.

For the second statement first assume $\phi = \mathrm{Ad}(u)$ to be inner. Then $x \mapsto u^{-1}x$ is the desired bimodule isomorphism. Conversely, if $f : \mathrm{X} \to {}_{\phi}\mathrm{X}_{\mathrm{Id}}$ is a bimodule isomorphism, then $f(x) = ux$ for some $u \in \mathcal{B}$ since $\mathcal{B} \cong \mathrm{End}_{\mathcal{A}}(\mathrm{X}_{\mathcal{A}})$. As the action of $\mathcal{B}$ is non-degenerate and f is invertible, u must also be invertible. Finally,

$$ubx = f(bx) = \phi(b)f(x) = \phi(b)ux \implies \phi(b) = ubu^{-1}.$$

□

In order to see how much $\mathrm{Aut}(\mathcal{A})$ contributes to $\mathrm{Pic}(\mathcal{A})$, we consider the map

$$l : \mathrm{Aut}(\mathcal{A}) \longrightarrow \mathrm{Pic}(\mathcal{A}), \quad l(\phi) = [{}_{\phi^{-1}}\mathcal{A}_{\mathrm{Id}}]. \tag{2.3}$$

A simple computation shows that, for $\phi, \psi \in \mathrm{Aut}(\mathcal{A})$, ${}_{\phi^{-1}}\mathcal{A}_{\mathrm{Id}} \otimes_{\mathcal{A}} {}_{\psi^{-1}}\mathcal{A}_{\mathrm{Id}} \cong {}_{\psi^{-1}\phi^{-1}}\mathcal{A}_{\mathrm{Id}}$. So the map l is a group homomorphism. By Proposition 2.3, we have an exact sequence of groups

$$1 \to \mathrm{InnAut}(\mathcal{A}) \to \mathrm{Aut}(\mathcal{A}) \xrightarrow{l} \mathrm{Pic}(\mathcal{A}). \tag{2.4}$$

Thus the image of l is a subgroup of $\mathrm{Pic}(\mathcal{A})$ isomorphic to $\mathrm{Aut}(\mathcal{A})/\mathrm{InnAut}(\mathcal{A})$, the *outer automorphisms* of $\mathcal{A}$.

2.3. **Remarks on the center.** Let $\mathcal{Z}(\mathcal{A})$ denote the center of $\mathcal{A}$. If $\mathrm{X} \in \underline{\mathrm{Pic}}(\mathcal{A})$, it is not generally true that $\mathcal{Z}(\mathcal{A})$ acts the same way on the left and right of X. Let $\underline{\mathrm{Pic}}_{\mathcal{Z}}(\mathcal{A})$ be the category of $(\mathcal{A}, \mathcal{A})$-invertible bimodules satisfying

$$zx = xz,$$

for all $x \in \mathrm{X}$ and $z \in \mathcal{Z}(\mathcal{A})$ and let $\mathrm{Pic}_{\mathcal{Z}}(\mathcal{A})$ be the corresponding group of isomorphism classes of objects.

Proposition 2.4. *There exists a group homomorphism* $h : \mathrm{Pic}(\mathcal{A}) \longrightarrow \mathrm{Aut}(\mathcal{Z}(\mathcal{A}))$ *so that*

$$1 \to \mathrm{Pic}_{\mathcal{Z}}(\mathcal{A}) \to \mathrm{Pic}(\mathcal{A}) \xrightarrow{h} \mathrm{Aut}(\mathcal{Z}(\mathcal{A})) \tag{2.5}$$

is exact. Moreover, if $\mathcal{A}$ *is commutative then* h *is split by the map* $l : \mathrm{Aut}(\mathcal{A}) \to \mathrm{Pic}(\mathcal{A})$.

Proof. Let $\mathrm{X} \in \underline{\mathrm{Pic}}(\mathcal{A})$. If $z \in \mathcal{Z}(\mathcal{A})$, then the map $x \mapsto xz$ is a bimodule endomorphism of X. Since $\mathrm{End}_{\mathcal{A}}(\mathrm{X}_{\mathcal{A}}) \cong \mathcal{A}$, there exists a unique $a = a(z) \in \mathcal{A}$ with $ax = xz$. It is clear that $a \in \mathcal{Z}(\mathcal{A})$ and that we have an induced algebra homomorphism

$$\phi_{\mathrm{X}} : \mathcal{Z}(\mathcal{A}) \longrightarrow \mathcal{Z}(\mathcal{A}), \quad z \mapsto a(z).$$

If $\mathrm{X}, \mathrm{Y} \in \underline{\mathrm{Pic}}(\mathcal{A})$ and $x \in \mathrm{X}, y \in \mathrm{Y}$, then

$$x \otimes yz = x \otimes \phi_{\mathrm{Y}}(z)y = x\phi_{\mathrm{Y}}(z) \otimes y = \phi_{\mathrm{X}}(\phi_{\mathrm{Y}}(z))x \otimes y.$$

Hence $\phi_{\mathrm{X}\otimes_{\mathcal{A}}\mathrm{Y}} = \phi_{\mathrm{X}}\phi_{\mathrm{Y}}$. Since $\phi_{\mathcal{A}} = \mathrm{Id}$, it follows that $\phi_{\mathrm{X}}\phi_{\mathrm{X}^*} = \phi_{\mathrm{X}^*}\phi_{\mathrm{X}} = \mathrm{Id}$ and ϕ_{X} is an automorphism of $\mathcal{Z}(\mathcal{A})$. Thus

$$h : \mathrm{Pic}(\mathcal{A}) \longrightarrow \mathrm{Aut}(\mathcal{A}), \quad \mathrm{X} \mapsto \phi_{\mathrm{X}}$$

is a homomorphism. The result now easily follows. □

Let us now assume that $\mathcal{A}$ is *commutative*. In this case, we regard $\mathrm{Aut}(\mathcal{A})$ as a subgroup of $\mathrm{Pic}(\mathcal{A})$ via the embedding $l : \mathrm{Aut}(\mathcal{A}) \to \mathrm{Pic}(\mathcal{A})$, $\phi \mapsto [{}_{\phi^{-1}}\mathcal{A}]$.

Corollary 2.5. *If $\mathcal{A}$ is commutative, then $\mathrm{Pic}_{\mathcal{A}}(\mathcal{A})$ is a normal subgroup of $\mathrm{Pic}(\mathcal{A})$, and $\mathrm{Pic}(\mathcal{A})$ is the semi-direct product of $\mathrm{Pic}_{\mathcal{A}}(\mathcal{A})$ and $\mathrm{Aut}(\mathcal{A})$. The induced product on $\mathrm{Aut}(\mathcal{A}) \times \mathrm{Pic}_{\mathcal{A}}(\mathcal{A})$ is given by*

$$(\phi, [\mathrm{X}]) \cdot (\psi, [\mathrm{Y}]) = (\phi \circ \psi, [{}_{\psi}\mathrm{X}_{\psi} \otimes_{\mathcal{A}} \mathrm{Y}]). \tag{2.6}$$

Proof. It follows from Proposition 2.4 that $\mathrm{Pic}_{\mathcal{A}}(\mathcal{A})$ is a normal subgroup of $\mathrm{Pic}(\mathcal{A})$, and Proposition 2.3 implies that the map

$$f : \mathrm{Aut}(\mathcal{A}) \times \mathrm{Pic}_{\mathcal{A}}(\mathcal{A}) \longrightarrow \mathrm{Pic}(\mathcal{A}), \quad f(\phi, [\mathrm{X}]) = [{}_{\phi^{-1}}\mathrm{X}],$$

is a bijection. Finally, note that

$${}_{\phi^{-1}}\mathrm{X} \otimes_{\mathcal{A}} {}_{\psi^{-1}}\mathrm{Y} \cong {}_{\phi^{-1}}\mathcal{A} \otimes_{\mathcal{A}} \mathrm{X} \otimes_{\mathcal{A}} {}_{\psi^{-1}}\mathcal{A} \otimes_{\mathcal{A}} \mathrm{Y} \cong {}_{\phi^{-1}}\mathcal{A} \otimes_{\mathcal{A}} \mathrm{X} \otimes_{\mathcal{A}} \mathcal{A}_{\psi} \otimes_{\mathcal{A}} \mathrm{Y}.$$

On the other hand,

$$\mathrm{X} \otimes_{\mathcal{A}} \mathcal{A}_{\psi} \cong \mathrm{X}_{\psi} \cong {}_{\psi^{-1}}\mathcal{A} \otimes_{\mathcal{A}} {}_{\psi}\mathrm{X}_{\psi}.$$

Therefore

$${}_{\phi^{-1}}\mathrm{X} \otimes_{\mathcal{A}} {}_{\psi^{-1}}\mathrm{Y} \cong {}_{\psi^{-1}\phi^{-1}}\mathcal{A} \otimes_{\mathcal{A}} {}_{\psi}\mathrm{X}_{\psi} \otimes_{\mathcal{A}} \mathrm{Y},$$

proving (2.6). □

For a commutative algebra $\mathcal{A}$, the group $\mathrm{Pic}_{\mathcal{A}}(\mathcal{A})$ is called the *classical* or *commutative* Picard group of $\mathcal{A}$.

Example 2.6. *Let M be a smooth manifold and let $\mathcal{A} = C^\infty(M)$ be the algebra of smooth complex-valued functions on M. Then*

$$\mathrm{Pic}_{\mathcal{A}}(\mathcal{A}) \cong \mathrm{Pic}(M) \cong H^2(M, \mathbb{Z}). \tag{2.7}$$

The first identification is a result of Serre-Swan's theorem (see e.g. [1]), where we identify line bundles with invertible bimodules by $L \mapsto \Gamma^\infty(L)$; the second is the Chern class map (see e.g. [19]). Hence $\mathrm{Pic}(\mathcal{A})$ is the semi-direct product of $H^2(M, \mathbb{Z})$ and $\mathrm{Diff}(M)$, where $\mathrm{Diff}(M)$ is the group of diffeomorphisms of M acting on $H^2(M, \mathbb{Z})$ by pull-back:

$$(\phi_1, l_1) \cdot (\phi_2, l_2) = (\phi_1 \circ \phi_2, \phi_2^*(l_1) + l_2), \quad \phi_1, \phi_2 \in \mathrm{Diff}(M),\ l_1, l_2 \in H^2(M, \mathbb{Z}).$$

3. Picard groups of deformed algebras

3.1. **Deformations of algebras and modules.** We will recall the basic definitions of formal deformations of associative algebras and modules over them, see [17].

Let $\mathcal{A}$ be a commutative and associative unital algebra over k, with $\mathbb{Q} \subseteq k$. We note that many of the definitions and results to follow hold for noncommutative algebras as well, but we will restrict ourselves to the commutative setting. Let $\mathcal{A}[[\lambda]]$ denote the space of formal power series with coefficients in $\mathcal{A}$ and formal parameter λ.

Definition 3.1. A formal *deformation* of $\mathcal{A}$ is a $k[[\lambda]]$-bilinear associative product $\star$ on $\mathcal{A}[[\lambda]]$ of the form

$$a_1 \star a_2 = \sum_{r=0}^{\infty} \lambda^r C_r(a_1, a_2), \quad a_1, a_1 \in \mathcal{A}, \tag{3.1}$$

where $C_r : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$ are bilinear maps, and $C_0(a_1, a_2) = a_1 a_2$. We also require that $1 \star a = a \star 1 = a$, for all $a \in \mathcal{A}$, where $1 \in \mathcal{A}$ is the unit element.

We denote the resulting deformed $k[[\lambda]]$-algebra by $\boldsymbol{\mathcal{A}} = (\mathcal{A}[[\lambda]], \star)$.

Definition 3.2. Two deformations $\boldsymbol{\mathcal{A}}_1 = (\mathcal{A}[[\lambda]], \star_1)$ and $\boldsymbol{\mathcal{A}}_2 = (\mathcal{A}[[\lambda]], \star_2)$ are *equivalent* if there are k-linear maps $T_r : \mathcal{A} \longrightarrow \mathcal{A}$ so that

$$T = \mathrm{Id} + \sum_{r=1}^{\infty} \lambda^r T_r : \ \boldsymbol{\mathcal{A}}_1 \longrightarrow \boldsymbol{\mathcal{A}}_2$$

is an algebra isomorphism.

The equivalence class of $\star$ is denoted by $[\star]$. The set of equivalence classes of deformations of $\mathcal{A}$ is denoted by $\mathrm{Def}(\mathcal{A})$. A deformation $\star = \sum_{r=0}^{\infty} \lambda^r C_r$ defines a Poisson bracket on $\mathcal{A}$ by

$$\{a_1, a_2\} := C_1(a_1, a_2) - C_1(a_2, a_1), \tag{3.2}$$

and equivalent deformations define equal Poisson brackets (see e.g. [9]). We denote the set of equivalence classes of deformations corresponding to a fixed Poisson bracket on $\mathcal{A}$ by $\mathrm{Def}(\mathcal{A}, \{\,,\,\})$.

We note that $\mathrm{Aut}(\mathcal{A})$ acts on formal deformations of $\mathcal{A}$ by $\star \mapsto \hat{\star} = \psi^*(\star)$, where

$$a_1 \mathbin{\hat{\star}} a_2 = \psi^{-1}(\psi(a_1) \star \psi(a_2)), \quad \psi \in \mathrm{Aut}(\mathcal{A}).$$

It is simple to check that two deformations $\boldsymbol{\mathcal{A}}_1$ and $\boldsymbol{\mathcal{A}}_2$ are isomorphic as $k[[\lambda]]$-algebras if and only if there exists a $\psi \in \mathrm{Aut}(\mathcal{A})$ with $[\psi^*(\star_1)] = [\star_2]$. Note, however, that this action of $\mathrm{Aut}(\mathcal{A})$ typically changes the Poisson bracket as defined in (3.2).

The following definition of a star product [2] provides our main example of formal deformations.

Definition 3.3. A *star product* on a manifold M is a formal deformation $\star = \sum_{r=0}^{\infty} \lambda^r C_r$ of $C^\infty(M)$ in the sense of Definition 3.1 so that each C_r is a bidifferential operator.

In this case, $\mathrm{Def}(M)$ denotes the moduli space of equivalence classes of star products on M; the subspace of equivalence classes of star products corresponding to a fixed Poisson structure π on M is denoted by $\mathrm{Def}(M, \pi)$. Here $\pi \in \Gamma^\infty(\bigwedge^2 TM)$ is the Poisson bivector field such that $\{f, g\} = \pi(df, dg)$ is the Poisson bracket, see e.g. [9, 31] for details on Poisson geometry. For star products, we adopt the convention

$$\{f, g\} := \frac{1}{i}(C_1(f, g) - C_1(g, f)), \quad f, g \in C^\infty(M). \tag{3.3}$$

We shall now consider module deformations of a right module $\mathrm{M}_{\mathcal{A}}$ over an arbitrary commutative unital $\mathcal{A}$.

Definition 3.4. A *deformation* of $\mathrm{M}_{\mathcal{A}}$ with respect to a deformation $\star$ of $\mathcal{A}$ (or, in short, a $\star$-module deformation of $\mathrm{M}_{\mathcal{A}}$) is a $k[[\lambda]]$-bilinear right $\boldsymbol{\mathcal{A}}$-module structure $\bullet$ on $\mathrm{M}[[\lambda]]$ of the form

$$m \bullet a = \sum_{r=0}^{\infty} \lambda^r R_r(m, a), \;\; m \in \mathrm{M}, \;\; a \in \mathcal{A}, \tag{3.4}$$

where each $R_r : \mathrm{M} \times \mathcal{A} \longrightarrow \mathrm{M}$ is bilinear and $R_0(m, a) = ma$ is the original module structure.

We denote the deformed module $(\mathrm{M}[[\lambda]], \bullet)$ by $\mathbf{M}_{\boldsymbol{\mathcal{A}}}$, and note that deformations of left modules are defined analogously.

Definition 3.5. Two $\star$-module deformations $\mathbf{M}_1$ and $\mathbf{M}_2$ of M are called *equivalent* if there exist k-linear maps $T_r : \mathrm{M} \longrightarrow \mathrm{M}$ so that

$$T = \mathrm{Id} + \sum_{r=1}^{\infty} \lambda^r T_r : \; \mathbf{M}_1 \longrightarrow \mathbf{M}_2 \tag{3.5}$$

is an $\boldsymbol{\mathcal{A}}$-module isomorphism.

As in the case of deformations of algebras, the automorphisms $\mathrm{Aut}(\mathrm{M}_{\mathcal{A}})$ act on $\star$-module deformations of $\mathrm{M}_{\mathcal{A}}$ by $\hat{\bullet} = \psi^*(\bullet)$, where

$$m \mathbin{\hat{\bullet}} a = \psi^{-1}(\psi(m) \bullet a), \quad m \in \mathrm{M}, \;\; a \in \mathcal{A}. \tag{3.6}$$

Two module deformations $\mathbf{M}_1$ and $\mathbf{M}_2$ of $\mathrm{M}_{\mathcal{A}}$ with respect to $\star$ are isomorphic if and only if there exists $\psi \in \mathrm{Aut}(\mathrm{M}_{\mathcal{A}})$ so that $\bullet_1$ is equivalent to $\psi^*(\bullet_2)$.

Recall from [7, 29] that if $\mathrm{M}_{\mathcal{A}}$ is finitely generated and projective, then module deformations always exist with respect to any deformation $\star$ and are still finitely generated and projective. Moreover, they are unique, up to equivalence. On the other hand, any finitely generated projective module over a deformed algebra $\boldsymbol{\mathcal{A}} = (\mathcal{A}[[\lambda]], \star)$ is isomorphic to a deformation of an $\mathcal{A}$-module with respect to $\star$. Hence the classical limit map

$$\mathrm{cl} : \boldsymbol{\mathcal{A}} \longrightarrow \mathcal{A}, \;\; \sum_{r=0}^{\infty} a_r \lambda_r \mapsto a_0$$

induces a map of K_0-groups

$$\mathrm{cl}_* : K_0(\boldsymbol{\mathcal{A}}) \longrightarrow K_0(\mathcal{A}), \quad [\mathbf{M}_{\boldsymbol{\mathcal{A}}}] \mapsto [\mathrm{M}_{\mathcal{A}}], \tag{3.7}$$

where $\mathbf{M}_{\boldsymbol{\mathcal{A}}} \cong (\mathrm{M}_{\mathcal{A}}[[\lambda]], \bullet)$, and this map is a group isomorphism [29, Thm. 4].

3.2. **Bimodule deformations and the classical-limit map.** Let ${}_{\mathcal{B}}\mathrm{X}_{\mathcal{A}}$ be a $(\mathcal{B},\mathcal{A})$-bimodule, and consider deformations $\boldsymbol{\mathcal{B}} = (\mathcal{B}[[\lambda]],\star')$ and $\boldsymbol{\mathcal{A}} = (\mathcal{A}[[\lambda]],\star)$.

Definition 3.6. A *bimodule deformation* of ${}_{\mathcal{B}}\mathrm{X}_{\mathcal{A}}$ with respect to $\star'$ and $\star$ (or, in short, a $(\star',\star)$-bimodule deformation of ${}_{\mathcal{B}}\mathrm{X}_{\mathcal{A}}$) is a $(\boldsymbol{\mathcal{B}},\boldsymbol{\mathcal{A}})$-bimodule structure on $\mathrm{X}[[\lambda]]$ (with right and left module structures being $k[[\lambda]]$-bilinear) of the form
(3.8)
$$b\bullet' x = \sum_{r=0}^{\infty}\lambda^r R'_r(b,x), \quad x\bullet a = \sum_{r=0}^{\infty}\lambda^r R_r(x,a), \quad \text{for } a\in\mathcal{A},\ b\in\mathcal{B} \ \text{ and } \ x\in \mathrm{X},$$
where $R'_r : \mathcal{B}\times\mathrm{X}\longrightarrow \mathrm{X}$ and $R_r : \mathrm{X}\times\mathcal{A}\longrightarrow\mathrm{X}$ are bilinear and so that $R'_0(b,x) = bx$ and $R_0(x,a) = xa$ are the original module structures.

We denote a $(\star',\star)$-bimodule deformation of ${}_{\mathcal{B}}\mathrm{X}_{\mathcal{A}}$ by $(\mathrm{X}[[\lambda]],\bullet',\bullet) = {}_{\boldsymbol{\mathcal{B}}}\mathbf{X}_{\boldsymbol{\mathcal{A}}}$. For a fixed deformation $\star$, we will later discuss conditions on $\star'$ guaranteeing the corresponding bimodule deformations to exist.

Analogously to the case of algebras and modules, bimodule automorphisms ψ of ${}_{\mathcal{B}}\mathrm{X}_{\mathcal{A}}$ act on $(\star',\star)$-bimodule deformations by $(\mathrm{X}[[\lambda]],\bullet',\bullet)\mapsto(\mathrm{X}[[\lambda]],\hat{\bullet}',\hat{\bullet})$, where
$$b\,\hat{\bullet}'\,x := \psi^{-1}(b\bullet'\psi(x)) \quad\text{and}\quad x\,\hat{\bullet}\,a := \psi^{-1}(\psi(x)\bullet a).$$
Two $(\star',\star)$-bimodule deformations of ${}_{\mathcal{B}}\mathrm{X}_{\mathcal{A}}$ are isomorphic if and only if they are equivalent up to this action.

Let us now assume that the deformations $\boldsymbol{\mathcal{A}} = (\mathcal{A}[[\lambda]],\star)$ and $\boldsymbol{\mathcal{B}} = (\mathcal{B}[[\lambda]],\star')$ are Morita equivalent.

Proposition 3.7. *Any object* $\mathbf{X}\in\underline{\mathrm{EBim}}(\boldsymbol{\mathcal{B}},\boldsymbol{\mathcal{A}})$ *is isomorphic to a* $(\star',\star)$*-bimodule deformation of some element* $\mathrm{X}\in\underline{\mathrm{EBim}}(\mathcal{B},\mathcal{A})$*, and this gives rise to a well-defined map*
$$\mathrm{cl}_* : \mathrm{EBim}(\boldsymbol{\mathcal{B}},\boldsymbol{\mathcal{A}})\longrightarrow\mathrm{EBim}(\mathcal{B},\mathcal{A}), \quad [\mathbf{X}]\mapsto[\mathrm{X}]. \tag{3.9}$$

Proof. Let ${}_{\boldsymbol{\mathcal{B}}}\mathbf{X}_{\boldsymbol{\mathcal{A}}}\in\underline{\mathrm{EBim}}(\boldsymbol{\mathcal{B}},\boldsymbol{\mathcal{A}})$. As a right $\boldsymbol{\mathcal{A}}$-module, $\mathbf{X}_{\boldsymbol{\mathcal{A}}}$ is isomorphic to $\boldsymbol{P}\star\boldsymbol{\mathcal{A}}^n$, for some full idempotent $\boldsymbol{P} = P + O(\lambda)\in M_n(\boldsymbol{\mathcal{A}})$ with an idempotent $P\in M_n(\mathcal{A})$. In particular, $\boldsymbol{\mathcal{B}}\cong\boldsymbol{P}\star M_n(\boldsymbol{\mathcal{A}})\star\boldsymbol{P}$.

Since $\boldsymbol{P}\star\boldsymbol{\mathcal{A}}^n\cong P\mathcal{A}^n[[\lambda]]$ as $k[[\lambda]]$-modules, see e.g. [7], we have an induced $(\boldsymbol{\mathcal{B}},\boldsymbol{\mathcal{A}})$-bimodule structure on $P\mathcal{A}^n[[\lambda]]$, which is a bimodule deformation of $\mathrm{Y} = P\mathcal{A}^n\in\underline{\mathrm{EBim}}(\mathcal{B},\mathcal{A})$ with respect to $\star'$ and $\star$ (recall that the fullness of $\boldsymbol{P}$ implies that P is automatically full). We denote this bimodule deformation by $\mathbf{Y}$.

Since $\mathbf{X}$ is isomorphic to $\mathbf{Y}$ as a right $\boldsymbol{\mathcal{A}}$-module, it follows from Proposition 2.3 that there exists $\psi = \psi_0 + O(\lambda)\in\mathrm{Aut}(\boldsymbol{\mathcal{B}})$ such that ${}_{\boldsymbol{\mathcal{B}}}\mathbf{X}_{\boldsymbol{\mathcal{A}}}$ is isomorphic to ${}_{\psi}\mathbf{Y}_{\boldsymbol{\mathcal{A}}}$, which is a deformation of ${}_{\psi_0}\mathrm{Y}_{\mathcal{A}}\in\underline{\mathrm{EBim}}(\mathcal{B},\mathcal{A})$. It is easy to check that the map $[\mathbf{X}]\mapsto[\mathrm{Y}]$ of isomorphism classes is well defined. $\square$

We remark that cl_* is not surjective in general, see Section 7.2.

In the case where $\mathcal{A}=\mathcal{B}$ and $\star=\star'$, a simple computation shows that if $\mathbf{X},\mathbf{X}'\in\underline{\mathrm{Pic}}(\boldsymbol{\mathcal{A}})$ are $(\star,\star)$-bimodule deformations of $\mathrm{X},\mathrm{X}'\in\underline{\mathrm{Pic}}(\mathcal{A})$, then
$$\mathrm{cl}_*([\mathbf{X}\otimes_{\boldsymbol{\mathcal{A}}}\mathbf{X}']) = [\mathrm{X}\otimes_{\mathcal{A}}\mathrm{X}].$$
This observation implies the next result.

Lemma 3.8. *The classical-limit map (3.9)*

$$\mathrm{cl}_* : \mathrm{Pic}(\boldsymbol{\mathcal{A}}) \longrightarrow \mathrm{Pic}(\mathcal{A}) \tag{3.10}$$

is a group homomorphism.

Example 3.9. *Consider the trivial deformation $\boldsymbol{\mathcal{A}}$ where the product is just the undeformed product of $\mathcal{A}$ extended $k[[\lambda]]$-bilinearily to $\boldsymbol{\mathcal{A}}$. Then*

$$\mathrm{Pic}(\boldsymbol{\mathcal{A}}) = \mathrm{Pic}_{\mathcal{A}}(\mathcal{A}) \ltimes \mathrm{Aut}(\boldsymbol{\mathcal{A}})$$

is larger than $\mathrm{Pic}(\mathcal{A})$ *due to the larger automorphism group, showing that (3.10) is not injective in general.*

Let us consider the group of self-equivalences of $\boldsymbol{\mathcal{A}}$,

$$\mathrm{Equiv}(\boldsymbol{\mathcal{A}}) := \{T \in \mathrm{Aut}(\boldsymbol{\mathcal{A}}) \mid T = \mathrm{Id} + O(\lambda)\}. \tag{3.11}$$

Note that, since $\mathcal{A}$ is commutative, $\mathrm{InnAut}(\boldsymbol{\mathcal{A}}) \subseteq \mathrm{Equiv}(\boldsymbol{\mathcal{A}})$ is a normal subgroup. We define the group of *outer self-equivalences* of $\boldsymbol{\mathcal{A}}$ as

$$\mathrm{OutEquiv}(\boldsymbol{\mathcal{A}}) := \frac{\mathrm{Equiv}(\boldsymbol{\mathcal{A}})}{\mathrm{InnAut}(\boldsymbol{\mathcal{A}})}. \tag{3.12}$$

Proposition 3.10. *For any $[\mathrm{X}] \in \mathrm{EBim}(\mathcal{B}, \mathcal{A})$ in the image of cl_*, $\mathrm{cl}_*^{-1}([\mathrm{X}])$ is in one-to-one correspondence with the outer self-equivalences of $\boldsymbol{\mathcal{B}}$.*

Proof. Let $\mathbf{X}$, $\mathbf{X}' \in \underline{\mathrm{EBim}}(\boldsymbol{\mathcal{B}}, \boldsymbol{\mathcal{A}})$, and let $[\mathrm{X}] = \mathrm{cl}_*([\mathbf{X}])$ and $[\mathrm{X}'] \in \mathrm{cl}_*([\mathbf{X}'])$. If $\mathrm{cl}_*([\mathbf{X}]) = \mathrm{cl}_*([\mathbf{X}'])$, then, in particular, $\mathrm{X}_{\mathcal{A}} \cong \mathrm{X}'_{\mathcal{A}}$ as right $\mathcal{A}$-modules. So, by uniqueness of right-module deformations (see e.g. [7]), $\mathbf{X}_{\boldsymbol{\mathcal{A}}} \cong \mathbf{X}'_{\boldsymbol{\mathcal{A}}}$ as right $\boldsymbol{\mathcal{A}}$-modules. Hence there exists a $\psi \in \mathrm{Aut}(\boldsymbol{\mathcal{B}})$ such that

$${}_{\psi}\mathbf{X}_{\boldsymbol{\mathcal{A}}} \cong {}_{\boldsymbol{\mathcal{B}}}\mathbf{X}'_{\boldsymbol{\mathcal{A}}}$$

as $(\boldsymbol{\mathcal{B}}, \boldsymbol{\mathcal{A}})$-bimodules.

Note that $\mathrm{cl}_*([{}_{\psi}\mathbf{X}_{\boldsymbol{\mathcal{A}}}]) = [{}_{\psi_0}\mathrm{X}_{\mathcal{A}}]$, where ψ_0 is such that $\psi = \psi_0 + O(\lambda)$. By Proposition 2.3, ${}_{\psi_0}\mathrm{X}_{\mathcal{A}} \cong {}_{\mathcal{A}}\mathrm{X}_{\mathcal{A}}$ if and only if $\psi_0 = \mathrm{Id}$. So, again by Proposition 2.3,

$$\mathrm{cl}_*^{-1}([\mathrm{X}]) = \{[{}_{\psi}\mathbf{X}_{\boldsymbol{\mathcal{A}}}] \mid \psi \in \mathrm{Equiv}(\boldsymbol{\mathcal{B}})\} \cong \mathrm{OutEquiv}(\boldsymbol{\mathcal{B}}).$$

□

For the group homomorphism $\mathrm{cl}_* : \mathrm{Pic}(\boldsymbol{\mathcal{A}}) \longrightarrow \mathrm{Pic}(\mathcal{A})$ from (3.10), we have $\ker(\mathrm{cl}_*) = \mathrm{cl}_*^{-1}([{}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}}])$. As a consequence, we obtain the following result.

Corollary 3.11. *For the classical-limit map (3.10), we have a group isomorphism* $\ker(\mathrm{cl}_*) \cong \mathrm{OutEquiv}(\boldsymbol{\mathcal{A}})$.

In fact, the group isomorphism is given by

$$\ker(\mathrm{cl}_*) \ni [{}_{\psi}\boldsymbol{\mathcal{A}}] \mapsto [\psi^{-1}] \in \mathrm{OutEquiv}(\boldsymbol{\mathcal{A}}),$$

see (2.3).

3.3. **Picard-group actions on deformations.** In this section, we will present a description of the image of the map cl_* in (3.10). Let $\mathcal{A}$ be a commutative unital algebra. We start with a slight extension of the discussion in [6].

Lemma 3.12. *Let $\star$ be a deformation of $\mathcal{A}$, and let X, Y be isomorphic objects in $\underline{\mathrm{Pic}}(\mathcal{A})$. Suppose $\boldsymbol{\mathcal{A}}' = (\mathcal{A}[[\lambda]], \star')$ and $\boldsymbol{\mathcal{A}}'' = (\mathcal{A}[[\lambda]], \star'')$ are deformations of $\mathcal{A}$ for which there exist a $(\star', \star)$-bimodule deformation $\mathbf{X} \in \underline{\mathrm{EBim}}(\boldsymbol{\mathcal{A}}', \boldsymbol{\mathcal{A}})$ and a $(\star'', \star)$-bimodule deformation $\mathbf{Y} \in \underline{\mathrm{EBim}}(\boldsymbol{\mathcal{A}}'', \boldsymbol{\mathcal{A}})$. Then $[\star'] = [\star'']$.*

Proof. Let $\psi_0 : \mathrm{X} \longrightarrow \mathrm{Y}$ be a bimodule isomorphism. By uniqueness of right-module deformations up to equivalence [7], there exists a right $\boldsymbol{\mathcal{A}}$-module isomorphism

$$\psi = \psi_0 + \sum_{r=1}^{\infty} \lambda^r \psi_r : \mathbf{X}_{\boldsymbol{\mathcal{A}}} \longrightarrow \mathbf{Y}_{\boldsymbol{\mathcal{A}}}.$$

Denote the left-module structure of $\mathbf{X}$ over $\boldsymbol{\mathcal{A}}'$ (resp. $\mathbf{Y}$ over $\boldsymbol{\mathcal{A}}''$) by $\bullet'$ (resp. $\bullet''$). For each $a \in \boldsymbol{\mathcal{A}}''$, the map

$$\mathbf{X} \ni x \ \mapsto \ \psi^{-1}(a \bullet'' \psi(x)) \in \mathbf{X}$$

is right $\boldsymbol{\mathcal{A}}$-linear. Hence there is a unique element $\phi(a) \in \boldsymbol{\mathcal{A}}'$ defined by

$$\phi(a) \bullet' x = \psi^{-1}(a \bullet'' \psi(x)), \quad \text{for all } x \in \mathbf{X}. \tag{3.13}$$

It is easy to check that $\phi = \phi_0 + \sum_{r=1}^{\infty} \lambda^r \phi_r : \boldsymbol{\mathcal{A}}'' \longrightarrow \boldsymbol{\mathcal{A}}'$ is an algebra isomorphism. Finally, notice that (3.13) in zeroth order implies that

$$\phi_0(a)x = \psi_0^{-1}(a\psi_0(x)) = ax, \quad \text{for all } a \in \mathcal{A}, \ x \in \mathrm{X}.$$

Hence $\phi_0 = \mathrm{Id}$, and $[\star'] = [\star'']$. ☐

Theorem 3.13. *There exists a natural action $\widetilde{\Phi} : \mathrm{Pic}(\mathcal{A}) \times \mathrm{Def}(\mathcal{A}) \longrightarrow \mathrm{Def}(\mathcal{A})$ so that two deformations $\star$ and $\star'$ of $\mathcal{A}$ are Morita equivalent if and only if $[\star]$ and $[\star']$ lie in the same $\widetilde{\Phi}$-orbit.*

Proof. Let $\boldsymbol{\mathcal{A}} = (\mathcal{A}[[\lambda]], \star)$ be a deformation of $\mathcal{A}$, and $\mathrm{X} \in \underline{\mathrm{Pic}}(\mathcal{A})$. We will first show that there is a deformation $\star'$ of $\mathcal{A}$ for which there exists a $(\star', \star)$-bimodule deformation of X.

Recall that, as a right $\mathcal{A}$-module, X can be identified with $P\mathcal{A}^n$ for some idempotent $P \in M_n(\mathcal{A})$. If $\boldsymbol{P} = P + O(\lambda) \in M_n(\boldsymbol{\mathcal{A}})$ is a (necessarily full) idempotent deforming P, then, by identifying $\boldsymbol{P} \star \boldsymbol{\mathcal{A}}^n$ with $\mathrm{X}[[\lambda]]$, we obtain an induced module deformation of $\mathrm{X}_{\mathcal{A}}$ with respect to $\star$. We denote this deformation by $\mathbf{X}_{\boldsymbol{\mathcal{A}}}$.

On the other hand, we can identify $\mathcal{A}[[\lambda]]$ with $\mathrm{End}(\mathbf{X}_{\boldsymbol{\mathcal{A}}})$ as $k[[\lambda]]$-modules, see e.g. [7], to obtain an induced deformation $(\mathcal{A}[[\lambda]], \hat{\star})$ of $\mathcal{A}$, together with a left action $\hat{\bullet}$ of $(\mathcal{A}[[\lambda]], \hat{\star})$ on $\mathbf{X}$. This turns $\mathbf{X}$ into an $(\hat{\star}, \star)$-bimodule. Note that, in general, the classical limit of $\mathbf{X}$ is isomorphic to a bimodule of the form ${}_{\psi}\mathrm{X}$, for some $\psi \in \mathrm{Aut}(\mathcal{A})$. Defining the deformation $\boldsymbol{\mathcal{A}}' = (\mathcal{A}[[\lambda]], \star')$, with

$$\star' := (\psi^{-1})^*(\hat{\star}),$$

we get a bimodule ${}_{\boldsymbol{\mathcal{A}}'}\mathbf{X}_{\boldsymbol{\mathcal{A}}}$ with classical limit ${}_{\mathcal{A}}\mathrm{X}_{\mathcal{A}}$, thereby obtaining a $(\star', \star)$-bimodule deformation of ${}_{\mathcal{A}}\mathrm{X}_{\mathcal{A}}$. It is also clear by the construction that ${}_{\boldsymbol{\mathcal{A}}'}\mathbf{X}_{\boldsymbol{\mathcal{A}}}$ is an equivalence bimodule.

A simple application of Lemma 3.12 shows that $[\star']$ depends only upon the isomorphism class $[\mathrm{X}] \in \mathrm{Pic}(\mathcal{A})$ and the equivalence class $[\star] \in \mathrm{Def}(\mathcal{A})$. So we have a well-defined map

$$\widetilde{\Phi} : \mathrm{Pic}(\mathcal{A}) \times \mathrm{Def}(\mathcal{A}) \longrightarrow \mathrm{Def}(\mathcal{A}), \quad [\star] \mapsto \widetilde{\Phi}_{\mathrm{X}}([\star]) = [\star'].$$

To see that $\widetilde{\Phi}$ defines an action, let $\star$ be a deformation of $\mathcal{A}$. Note that for the identity $\mathcal{A} \in \underline{\mathrm{Pic}}(\mathcal{A})$, $\widetilde{\Phi}_{\mathcal{A}} = \mathrm{Id}$, since $\boldsymbol{\mathcal{A}}$ is itself an $(\boldsymbol{\mathcal{A}}, \boldsymbol{\mathcal{A}})$-equivalence bimodule deforming $\mathcal{A}$. Now let $\mathrm{X}, \mathrm{Y} \in \underline{\mathrm{Pic}}(\mathcal{A})$, and suppose $\star' \in \widetilde{\Phi}_{\mathrm{X}}([\star])$ and $\star'' \in \widetilde{\Phi}_{\mathrm{Y}}([\star'])$, in such a way that

$$[\star''] = \widetilde{\Phi}_{\mathrm{Y}} \widetilde{\Phi}_{\mathrm{X}}([\star]).$$

Then there exists a $(\star', \star)$-bimodule deformation of X and a $(\star'', \star')$-bimodule deformation of Y, denoted by $\mathbf{X}$ and $\mathbf{Y}$, respectively. But $\mathbf{Y} \otimes_{\boldsymbol{\mathcal{A}}'} \mathbf{X}$ is a $(\star'', \star)$-bimodule deformation of $\mathrm{Y} \otimes_{\mathcal{A}} \mathrm{X}$. So

$$[\star''] = \widetilde{\Phi}_{\mathrm{X} \otimes_{\mathcal{A}} \mathrm{Y}}([\star]).$$

Finally, if $\star'$ and $\star$ are Morita equivalent, there exists a corresponding equivalence bimodule $\mathbf{X}$. If $[\mathrm{X}] = \mathrm{cl}_*([\mathbf{X}])$, then $[\star'] = \widetilde{\Phi}_{\mathrm{X}}([\star])$. □

We observe that, for a deformation $\star$ of $\mathcal{A}$, the image of the map cl_* in (3.10) is the isotropy group of $\widetilde{\Phi}$ at $[\star]$.

It turns out that the action $\widetilde{\Phi}$ can be written as a combination of its restriction to the subgroup $\mathrm{Pic}_{\mathcal{A}}(\mathcal{A})$ and the natural action of $\mathrm{Aut}(\mathcal{A})$ on deformations of $\mathcal{A}$, see [6].

Proposition 3.14. *There exists a natural action $\Phi : \mathrm{Pic}_{\mathcal{A}}(\mathcal{A}) \times \mathrm{Def}(\mathcal{A}) \longrightarrow \mathrm{Def}(\mathcal{A})$ such that two deformations $\star, \star'$ of $\mathcal{A}$ are Morita equivalent if and only if there exists an automorphism ψ of $\mathcal{A}$ so that $[\psi^*(\star')]$ and $[\star]$ lie in the same Φ-orbit.*

Proof. The action is just the one from Theorem 3.13, restricted to $\mathrm{Pic}_{\mathcal{A}}(\mathcal{A})$. So, if $[\psi^*(\star')]$ and $[\star]$ lie in the same Φ-orbit, then $\psi^*(\star')$ and $\star$ are Morita equivalent. Since $\star'$ and $\psi^*(\star')$ are isomorphic deformations, it follows that $\star'$ is also Morita equivalent to $\star$.

On the other hand, if $\star'$ and $\star$ are Morita equivalent, then there exists a corresponding equivalence bimodule $\mathbf{X}$, and $\mathrm{cl}_*([\mathbf{X}]) = [{}_{\phi}\mathrm{X}]$ for some $\phi \in \mathrm{Aut}(\mathcal{A})$ and $\mathrm{X} \in \underline{\mathrm{Pic}}_{\mathcal{A}}(\mathcal{A})$. Replacing $\star'$ by $\star'' = (\phi^{-1})^*(\star')$, we obtain a $(\star'', \star)$-bimodule deformation of X. So $[\psi^*(\star')] = \Phi_{\mathrm{X}}([\star])$, where $\psi = \phi^{-1}$. □

As observed in [6], the restricted action Φ has the nice feature that deformations in the same Φ-orbit correspond to the same Poisson bracket. So if we fix a Poisson bracket $\{\ ,\ \}$ on $\mathcal{A}$, we have the following result.

Corollary 3.15. *There is a natural action $\Phi : \mathrm{Pic}_{\mathcal{A}}(\mathcal{A}) \times \mathrm{Def}(\mathcal{A}, \{\ ,\ \}) \longrightarrow \mathrm{Def}(\mathcal{A}, \{\ ,\ \})$ in such a way that two deformations $\star, \star'$ of $(\mathcal{A}, \{\ ,\ \})$ are Morita equivalent if and only if there exists a Poisson automorphism ψ of $\mathcal{A}$ so that $[\psi^*(\star')]$ and $[\star]$ lie in the same Φ-orbit.*

Remark 3.16. In fact, one can compute the semiclassical limit of the action Φ, obtaining a first-order obstruction to Morita equivalence of formal deformations in terms of algebraic Poisson cohomology [20]; the arguments are similar to the case of star products [6]. A detailed analysis will be presented in a future work.

Remark 3.17. Consider again $\mathcal{A} = C^\infty(M)$, in which case $\mathrm{Pic}_{\mathcal{A}}(\mathcal{A}) \cong \mathrm{Pic}(M) \cong H^2(M,\mathbb{Z})$, see Example 2.6. For a star product $\star$ on M and $[L] \in \mathrm{Pic}(M)$, one can show that there exists a $\star' \in \Phi_L([\star])$ which is again a star product on M, i.e. a deformation given by bidifferential cochains, see [6, 7]. So the action Φ in Proposition 3.14 restricts to an action

$$\Phi : \mathrm{Pic}(M) \times \mathrm{Def}(M) \longrightarrow \mathrm{Def}(M), \quad [\star] \mapsto \Phi_L([\star]). \tag{3.14}$$

For a fixed Poisson structure π on M, two star products $\star$ and $\star'$ on (M,π) are Morita equivalent if and only if there exists a Poisson diffeomorphism $\psi : M \longrightarrow M$ such that $[\psi^*(\star')]$ and $[\star]$ lie in the same Φ-orbit.

Let us fix a deformation $\star$ of $\mathcal{A}$, corresponding to a Poisson bracket $\{\,,\,\}$. In order to describe the image of $\mathrm{cl}_* : \mathrm{Pic}(\boldsymbol{\mathcal{A}}) \longrightarrow \mathrm{Pic}(\mathcal{A})$, we may first consider the map

$$\mathrm{cl}_*^r = \mathrm{pr} \circ \mathrm{cl}_* : \ \mathrm{Pic}(\boldsymbol{\mathcal{A}}) \longrightarrow \mathrm{Pic}_{\mathcal{A}}(\mathcal{A}), \tag{3.15}$$

where $\mathrm{pr} : \mathrm{Pic}(\mathcal{A}) \longrightarrow \mathrm{Pic}_{\mathcal{A}}(\mathcal{A})$ is the natural projection, see Corollary 2.5.

Note that, by Proposition 3.14, the image of cl_*^r is given by those $[\mathrm{X}] \in \mathrm{Pic}_{\mathcal{A}}(\mathcal{A})$ for which there exists a $\star' \in \Phi_{\mathrm{X}}([\star])$ and $\psi \in \mathrm{Aut}(\mathcal{A})$ such that $[\star] = [\psi^*(\star')]$. Since $\star'$ and $\star$ correspond to the same Poisson bracket $\{\,,\}$, it follows that ψ must be a Poisson map. As a result, we have the following corollary.

Corollary 3.18. *The image of the map* cl_*^r *in (3.15) is given by*

$$\mathrm{im}(\mathrm{cl}_*^r) = \{[\mathrm{X}] \in \mathrm{Pic}_{\mathcal{A}}(\mathcal{A}) \mid \exists \psi \in \mathrm{Poiss}(\mathcal{A}, \{\,,\,\}) \textit{ and } \star' \in \Phi_{\mathrm{X}}([\star]) \textit{ with } [\psi^*(\star')] = [\star]\},$$

where $\mathrm{Poiss}(\mathcal{A}, \{\,,\,\})$ *denotes the group of Poisson automorphisms of* $(\mathcal{A}, \{\,,\,\})$.

In terms of the image of cl_*^r, the image of cl_* can be described as follows. For $[\mathrm{X}] \in \mathrm{im}(\mathrm{cl}_*^r)$, let

$$P_{\mathrm{X}} := \{\psi \in \mathrm{Poiss}(\mathcal{A}, \{\,,\,\}) \text{ for which } \exists \star' \in \Phi_{\mathrm{X}}([\star]) \text{ with } [\psi^*(\star')] = [\star]\}.$$

Corollary 3.19. *The image of* cl_* *is given by*

$$\mathrm{im}(\mathrm{cl}_*) = \{[{}_\psi\mathrm{X}] \mid [\mathrm{X}] \in \mathrm{im}(\mathrm{cl}_*^r) \textit{ and } \psi \in P_{\mathrm{X}}\}.$$

Note that, since $[{}_\psi\mathrm{X}] \in \mathrm{im}(\mathrm{cl}_*)$ implies that $\psi \in \mathrm{Poiss}(\mathcal{A}, \{\,,\})$ and $\mathrm{Pic}(\mathcal{A}) = \mathrm{Aut}(\mathcal{A}) \ltimes \mathrm{Pic}_{\mathcal{A}}(\mathcal{A})$ (see Corollary 2.5), it immediately follows that cl_* is not onto in general. We will come back to this question in Section 7.2.

4. Bimodule deformations and their semiclassical limit

In this section, we identify the semiclassical limit of bimodule deformations and discuss the corresponding deformation quantization problem.

4.1. **Contravariant connections.** Let $\mathcal{A}$ be a commutative unital k-algebra, and let $\{\, ,\, \}$ be a Poisson bracket on $\mathcal{A}$. The next definition follows the notion of contravariant derivatives in Poisson geometry [14, 20, 31].

Definition 4.1. A *contravariant connection* on $\mathrm{X} \in \underline{\mathrm{Pic}}_{\mathcal{A}}(\mathcal{A})$ is a k-bilinear map $D : \mathcal{A} \times \mathrm{X} \longrightarrow \mathrm{X}$ satisfying

i) $D(ab, x) = D(a, x)b + D(b, x)a$,
ii) $D(a, bx) = bD(a, x) + \{a, b\}x$.

We denote $D(a, \cdot) = D_a$. The *curvature* of D is the skew-symmetric bilinear map $\mathrm{curv}(D) : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$ defined by

$$\mathrm{curv}(D)(a, b) = D_a D_b - D_b D_a - D_{\{a,b\}}, \quad a, b \in \mathcal{A}.$$

Here, we are using the identification of $\mathrm{End}(\mathrm{X})$ with $\mathcal{A}$. As usual, D is called *flat* if $\mathrm{curv}(D) = 0$.

Note that if D and D' are contravariant connections on X, then their difference $D - D'$ is a derivation of $\mathcal{A}$, in the sense that $D_a x - D'_a x = \alpha(a)x$ determines a derivation α of $\mathcal{A}$. Conversely, if $\alpha : \mathcal{A} \longrightarrow \mathcal{A}$ is a derivation and D is a contravariant connection, so is $D + \alpha$.

Example 4.2 ([14, 31])**.** *Let (M, π) be a Poisson manifold, L a line bundle over M and ∇ any (ordinary) connection on L. For $f \in C^\infty(M)$, let X_f denote its Hamiltonian vector field. Then*

$$D : C^\infty(M) \times \Gamma^\infty(L) \longrightarrow \Gamma^\infty(L)$$

defined by $D(f, s) = \nabla_{X_f} s$ is a contravariant connection. When π is symplectic, any contravariant connection arises in this way.

Example 4.3. *For the $(\mathcal{A}, \mathcal{A})$-bimodule $\mathcal{A}$, with respect to left and right multiplication, we can define the canonical contravariant connection d by*

$$d_a(b) = \{a, b\}, \quad a, b \in \mathcal{A}.$$

This connection is flat as a consequence of the Jacobi identity of $\{\, ,\, \}$.

Let $\mathrm{X}, \mathrm{X}' \in \underline{\mathrm{Pic}}_{\mathcal{A}}(\mathcal{A})$ be equipped with contravariant connections D and D', respectively.

Definition 4.4. An *isomorphism of bimodules preserving contravariant connections* is a bimodule isomorphism $T : \mathrm{X} \longrightarrow \mathrm{X}'$ satisfying

$$T(D_a x) = D'_a T x,$$

for all $a \in \mathcal{A}$ and $x \in \mathrm{X}$.

Note that isomorphic contravariant connections have the same curvature. We denote the set of contravariant connections on X with curvature τ by $\underline{\mathrm{Conn}}(\mathrm{X}, \tau)$; the set of isomorphism classes of elements in $\underline{\mathrm{Conn}}(\mathrm{X}, \tau)$ is denoted by $\mathrm{Conn}(\mathrm{X}, \tau)$.

If α is a derivation of $\mathcal{A}$, we call it a *Poisson derivation* if

$$\alpha(\{a, b\}) = \{\alpha(a), b\} + \{a, \alpha(b)\}.$$

We denote the set of derivations and Poisson derivations of $\mathcal{A}$ by $\mathrm{Der}(\mathcal{A})$ and $\mathrm{PDer}(\mathcal{A})$, respectively.

Lemma 4.5. *Let D be a contravariant connection on $\mathrm{X} \in \underline{\mathrm{Pic}}_{\mathcal{A}}(\mathcal{A})$, and let α be a derivation of $\mathcal{A}$. Then $\mathrm{curv}(D+\alpha) = \mathrm{curv}(D)$ if and only if $\alpha \in \mathrm{PDer}(\mathcal{A})$. In particular, $\underline{\mathrm{Conn}}(\mathrm{X}, \tau)$ is an affine space over k modeled on $\mathrm{PDer}(\mathcal{A})$.*

Proof. If we expand the equation

$$(D+\alpha)_a(D+\alpha)_b(x) - (D+\alpha)_b(D+\alpha)_a(x) - (D+\alpha)_{\{a,b\}}(x) = \tau,$$

it is simple to check that $\mathrm{curv}(D) = \tau$ if and only if α is a Poisson derivation of $\mathcal{A}$. $\square$

Proposition 4.6. *Let $\mathrm{X} \in \underline{\mathrm{Pic}}_{\mathcal{A}}(\mathcal{A})$, $D \in \underline{\mathrm{Conn}}(\mathrm{X}, \tau)$, and $\alpha \in \mathrm{Der}(\mathcal{A})$. Then $(\mathrm{X}, D+\alpha)$ is isomorphic to (X, D) if and only if $(\mathcal{A}, d+\alpha)$ is isomorphic to $(\mathcal{A}, d)$.*

Proof. Suppose there exists a bimodule automorphism $\phi : \mathcal{A} \longrightarrow \mathcal{A}$ satisfying

$$\phi\, d_a = (d_a + \alpha(a))\, \phi,$$

for all $a \in \mathcal{A}$. For the bimodule with contravariant connection (X, D), consider the tensor products $(\mathrm{X} \otimes \mathcal{A}, D \otimes d)$ and $(\mathrm{X} \otimes \mathcal{A}, D \otimes (d+\alpha))$, where the tensor product of two connections is defined in the natural way:

$$D_1 \otimes D_2(x_1 \otimes x_2) = D_1(x_1) \otimes x_2 + x_1 \otimes D_2(x_2).$$

One can check that the natural identification $\mathrm{X} \otimes \mathcal{A} \longrightarrow \mathrm{X}$, $x \otimes a \mapsto xa$ induces isomorphisms

$$(\mathrm{X} \otimes_{\mathcal{A}} \mathcal{A}, D \otimes d) \xrightarrow{\sim} (\mathrm{X}, D),$$

and

$$(\mathrm{X} \otimes_{\mathcal{A}} \mathcal{A}, D \otimes (d+\alpha)) \xrightarrow{\sim} (\mathrm{X}, D+\alpha).$$

Finally, note that ϕ gives rise to a map $\mathrm{X} \otimes_{\mathcal{A}} \mathcal{A} \longrightarrow \mathrm{X} \otimes_{\mathcal{A}} \mathcal{A}$ through $x \otimes a \mapsto x \otimes \phi(a)$, which establishes an isomorphism between $D \otimes d$ and $D \otimes (d+\alpha)$. Hence there is an induced isomorphism

$$(\mathrm{X}, D) \xrightarrow{\sim} (\mathrm{X}, D+\alpha).$$

For the converse, suppose that there is an isomorphism

$$\psi : (\mathrm{X}, D) \longrightarrow (\mathrm{X}, D+\alpha).$$

Define the dual module with contravariant connection (X^*, D^*), where D^* and D are related in the usual way:

$$d_a\langle x^*, x\rangle = \langle D^*_a x^*, x\rangle + \langle x^*, D_a x\rangle,$$

where $a \in \mathcal{A}$, $x \in \mathrm{X}$, $x^* \in \mathrm{X}^* = \mathrm{Hom}(\mathrm{X}, \mathcal{A})$ and $\langle\ ,\ \rangle$ is the pairing between X and X^*. Then ψ induces an isomorphism

$$(\mathrm{X} \otimes_{\mathcal{A}} \mathrm{X}^*, D \otimes D^*) \xrightarrow{\sim} (\mathrm{X} \otimes_{\mathcal{A}} \mathrm{X}^*, (D+\alpha) \otimes D^*).$$

Finally, notice that the isomorphism $\mathrm{X} \otimes_{\mathcal{A}} \mathrm{X}^* \longrightarrow \mathcal{A}$ given by $x \otimes x^* \mapsto \langle x^*, x\rangle$ induces isomorphisms $(\mathrm{X} \otimes_{\mathcal{A}} \mathrm{X}^*, D \otimes D^*) \xrightarrow{\sim} (\mathcal{A}, d)$ and $(\mathrm{X} \otimes_{\mathcal{A}} \mathrm{X}^*, (D+\alpha) \otimes D^*) \xrightarrow{\sim} (\mathcal{A}, d+\alpha)$. Therefore $(\mathcal{A}, d)$ and $(\mathcal{A}, d+\alpha)$ are isomorphic. $\square$

Corollary 4.7. *Isomorphism classes of contravariant connections on* X *with fixed curvature* τ *are classified by isomorphism classes of flat contravariant connections on* $\mathcal{A}$.

We have the following characterization of Poisson derivations yielding isomorphic connections.

Lemma 4.8. *A contravariant connection* $D+\alpha$ *on* X *is isomorphic to* D *if and only if there exists an invertible element* $u \in \mathcal{A}$ *such that* $\alpha = u^{-1}\{\cdot, u\}$.

Proof. By Corollary 4.7, it suffices to consider the case $\mathrm{X} = \mathcal{A}$, $D = d$. Any bimodule automorphism of $\mathcal{A}$ is given by multiplication by an invertible element $u \in \mathcal{A}$. Note that $u(d+\alpha)_a b = d_a b$ for all $a, b \in \mathcal{A}$, if and only if

$$u(\{a, b\} + \alpha(a)b) = \{a, ub\} = \{a, b\}u + \{a, u\}b,$$

which is equivalent to $\alpha(a) = \{a, u\}/u$. □

Definition 4.9. Let $u \in \mathcal{A}$ be invertible. We call the derivations of the form $\alpha = u^{-1}\{u, \cdot\}$ *integral derivations.* We denote the abelian group of integral derivations of $\mathcal{A}$ by $\mathrm{IDer}(\mathcal{A})$.

Note that if $uv = 1$, then $u^{-1}\{u, \cdot\} = v^{-1}\{\cdot, v\}$, so we have two alternative definitions of integral derivations. Note also that the addition $\alpha + \alpha'$ in $\mathrm{IDer}(\mathcal{A})$ corresponds to the product uu'. Heuristically speaking, α corresponds to the inner Poisson derivation with the (not necessarily existing) logarithm of u.

Since isomorphic connections have the same curvature, it follows from Lemma 4.5 that $\mathrm{IDer}(\mathcal{A}) \subseteq \mathrm{PDer}(\mathcal{A})$. Of course, this can also be checked directly.

Corollary 4.10. *The set* $\mathrm{Conn}(\mathcal{A}, 0)$ *is an affine space over* $\mathbb{Z}$ *modeled on* $\mathrm{PDer}(\mathcal{A})/\mathrm{IDer}(\mathcal{A})$ *with canonical origin* $[d]$. *Hence the map*

$$\mathrm{Conn}(\mathcal{A}, 0) \longrightarrow \mathrm{PDer}(\mathcal{A})/\mathrm{IDer}(\mathcal{A}), \quad [D] \mapsto [D-d]$$

is a bijection. By Corollary 4.7, $\mathrm{Conn}(\mathrm{X}, \tau)$ *is also an affine space over* $\mathbb{Z}$ *modeled on* $\mathrm{PDer}(\mathcal{A})/\mathrm{IDer}(\mathcal{A})$, *but with no canonical origin in general.*

Example 4.11. *Let* (M, ω) *be a symplectic manifold and* $\mathcal{A} = C^\infty(M)$, *equipped with the induced Poisson bracket. We have*

$$\mathrm{IDer}(\mathcal{A}) \cong \{f^{-1}df \mid f : M \to \mathbb{C}^*\} \subseteq Z^1(M, \mathbb{C}),$$

where $Z^1(M, \mathbb{C})$ *denotes the space of closed complex-valued* 1*-forms on* M.

Recall that on a contractible open set $\mathcal{O}_\alpha \subseteq M$, *any smooth* $f : M \longrightarrow \mathbb{C}^*$ *can be written as* $f = \mathrm{e}^{2\pi i c_\alpha}$, *for* $c_\alpha \in C^\infty(\mathcal{O}_\alpha)$. *Hence* $f^{-1}df = 2\pi i dc_\alpha$ *and* $c_\alpha - c_\beta \in \mathbb{Z}$ *on overlaps* $\mathcal{O}_\alpha \cap \mathcal{O}_\beta$. *It follows that*

$$\mathrm{pr}(\mathrm{IDer}(\mathcal{A})) = 2\pi i H^1_{\mathrm{dR}}(M, \mathbb{Z}),$$

where $\mathrm{pr} : Z^1(M, \mathbb{C}) \longrightarrow H^1_{\mathrm{dR}}(M, \mathbb{C})$ *denotes the usual projection and* $H^1_{\mathrm{dR}}(M, \mathbb{Z})$ *is the image of* $H^1(M, \mathbb{Z})$ *under the natural map*

$$H^1(M, \mathbb{Z}) \longrightarrow H^1_{\mathrm{dR}}(M, \mathbb{C}).$$

This motivates our terminology of integral derivations. Note that

$$\mathrm{PDer}(\mathcal{A})/\mathrm{IDer}(\mathcal{A}) \cong Z^1(M,\mathbb{C})/\{f^{-1}df \mid f : M \to \mathbb{C}^*\},$$

and since $\{f^{-1}df \mid f : M \to \mathbb{C}^*\}$ *contains all exact* 1*-forms, there is a well-defined surjective map*

$$H^1_{\mathrm{dR}}(M,\mathbb{C}) \to \frac{Z^1(M,\mathbb{C})}{\{f^{-1}df \mid f : M \to \mathbb{C}^*\}}.$$

The kernel of this map is just $\mathrm{pr}(\mathrm{IDer}(\mathcal{A})) = 2\pi i H^1_{\mathrm{dR}}(M,\mathbb{Z})$*. So, as abelian groups,*

$$\frac{\mathrm{PDer}(\mathcal{A})}{\mathrm{IDer}(\mathcal{A})} \cong \frac{H^1_{\mathrm{dR}}(M,\mathbb{C})}{2\pi i H^1_{\mathrm{dR}}(M,\mathbb{Z})}. \tag{4.1}$$

Corollary 4.10 implies the well-known fact that isomorphism classes of lines bundles with connections of fixed curvature are classified by the quotient (4.1).

Example 4.12. *Let* (M,π) *be a general Poisson manifold. In this case, integral Poisson derivations are Poisson vector fields of the form*

$$f^{-1}\{\cdot, f\} = f^{-1}d_\pi f,$$

for $f \in C^\infty(M)$*, where* $d_\pi = [\pi,\cdot\,]$ *and* $[\ ,\ ]$ *is the Schouten bracket* [9, 31]*. Consider the map*

$$\tilde{\pi} : \Omega^1(M) \to \chi^1(M), \quad \tilde{\pi}(\beta) = \pi(\cdot,\beta).$$

Since $d_\pi f = \tilde{\pi}(df)$*, there is an induced map in cohomology* $\pi^* : H^1_{\mathrm{dR}}(M,\mathbb{C}) \longrightarrow H^1_\pi(M,\mathbb{C})$*, where* $H^1_\pi(M,\mathbb{C})$ *denotes the first Poisson cohomology of* M*, see e.g.* [9]*. For* $f \in C^\infty(M)$*, we have*

$$f^{-1}d_\pi f = \tilde{\pi}(f^{-1}df),$$

and hence

$$\{[f^{-1}d_\pi f]_\pi,\ \ f : M \to \mathbb{C}^*\} = \{\pi^*[f^{-1}df],\ \ f : M \to \mathbb{C}^*\} = 2\pi i H^1_\pi(M,\mathbb{Z}),$$

where $H^1_\pi(M,\mathbb{Z}) := \pi^* H^1_{\mathrm{dR}}(M,\mathbb{Z})$*. Analogously to the previous example, we conclude that*

$$\frac{\mathrm{PDer}(\mathcal{A})}{\mathrm{IDer}(\mathcal{A})} \cong \frac{H^1_\pi(M,\mathbb{C})}{2\pi i H^1_\pi(M,\mathbb{Z})} \tag{4.2}$$

as abelian groups. Corollary 4.10 asserts the well-known fact that (4.2) classifies isomorphism classes of line bundles over M *with contravariant connections of fixed curvature.*

4.2. **The semiclassical limit of bimodule deformations.** Let $\mathrm{X} \in \underline{\mathrm{Pic}}_{\mathcal{A}}(\mathcal{A})$. Let $\star$ be a deformation of $(\mathcal{A},\{\ ,\ \})$, and let $\star' \in \Phi_{\mathrm{X}}([\star])$. We write these deformations as

$$\star = \sum_{r=0}^{\infty} \lambda^r C_r \quad \text{and} \quad \star' = \sum_{r=0}^{\infty} \lambda^r C'_r. \tag{4.3}$$

Definition 4.13. We define the skew-symmetric bilinear map $\tau : \mathcal{A}\times\mathcal{A} \longrightarrow \mathcal{A}$, corresponding to $\star', \star$, by

$$\tau(a_1,a_2) = (C_2 - C'_2)(a_1,a_2) - (C_2 - C'_2)(a_2,a_1), \quad a_1, a_2 \in \mathcal{A}.$$

Since $\mathcal{A}$ is commutative and $\star$ and $\star'$ deform the same Poisson bracket, τ can also be written as

$$\tau(a_1, a_2) = \left(\frac{1}{\lambda^2} \left([a_1, a_2]_\star - [a_1, a_2]_{\star'} \right) \right) \Big|_{\lambda=0} \tag{4.4}$$

By Proposition 3.14 the algebras $\boldsymbol{\mathcal{A}} = (\mathcal{A}[[\lambda]], \star)$ and $\boldsymbol{\mathcal{A}}' = (\mathcal{A}[[\lambda]], \star')$ are Morita equivalent. Let

$$\mathbf{X} = (\mathrm{X}[[\lambda]], \bullet', \bullet) \in \underline{\mathrm{EBim}}(\boldsymbol{\mathcal{A}}', \boldsymbol{\mathcal{A}})$$

be a $(\star', \star)$-bimodule deformation of X, and let us write

$$\bullet' = \sum_{r=0}^{\infty} \lambda^r R'_r \quad \text{and} \quad \bullet = \sum_{r=0}^{\infty} \lambda^r R_r.$$

Proposition 4.14. *Suppose $C_1 = C'_1$. Then the map*

$$D = R_1 - R'_1 : \mathcal{A} \times \mathrm{X} \longrightarrow \mathrm{X}$$

is a contravariant connection, and $\mathrm{curv}(D) = \tau$.

Remark 4.15. In the case of star products, in order to be consistent with the convention (3.3), we define $D = (R_1 - R'_1)/i$; in this case $\mathrm{curv}(D) = -\tau$.

Proof. The proof is analogous to the proofs of [6, Prop. 4.3 and Thm 5.3] in the case of star products. We consider the equations relating $\star'$, $\star$, $\bullet'$ and $\bullet$:

$$(a_1 \star' a_2) \bullet' x = a_1 \bullet' (a_2 \bullet' x), \tag{4.5}$$

$$x \bullet (a_1 \star a_2) = (x \bullet a_1) \bullet a_2, \tag{4.6}$$

$$(a_1 \bullet' x) \bullet a_2 = a_1 \bullet' (x \bullet a_2), \tag{4.7}$$

for $a_1, a_2 \in \mathcal{A}$, $x \in \mathrm{X}$. A suitable combination of equations (4.5), (4.6) and (4.7) in order λ shows that $D = R_1 - R'_1$ is a contravariant connection. A combination of these equations in order λ^2 implies that $\mathrm{curv}(D) = \tau$, see [6] for details. □

In particular, $\mathrm{curv}(D)$ depends only upon $\star'$ and $\star$, and not on any particular $(\star', \star)$-bimodule deformation of X.

Let $\underline{\mathrm{Def}}(\mathrm{X}, \star', \star)$ be the set of $(\star', \star)$-bimodule deformations of X; the set of isomorphism classes of elements in $\underline{\mathrm{Def}}(\mathrm{X}, \star', \star)$ is denoted by $\mathrm{Def}(\mathrm{X}, \star', \star)$.

Using Proposition 4.14, we define the semiclassical limit map

$$S : \underline{\mathrm{Def}}(\mathrm{X}, \star', \star) \longrightarrow \underline{\mathrm{Conn}}(\mathrm{X}, \tau), \quad (\mathrm{X}[[\lambda]], \bullet', \bullet) \mapsto D = R_1 - R'_1. \tag{4.8}$$

Example 4.16. *Consider the $(\mathcal{A}, \mathcal{A})$-bimodule $\mathcal{A}$ with respect to left and right multiplication. In this case, the contravariant connection $S(\boldsymbol{\mathcal{A}})$ associated to the $(\star, \star)$-bimodule deformation of $\mathcal{A}$ given by $\boldsymbol{\mathcal{A}}$ is just d of Example 4.3,*

$$d_a(b) = C_1(a, b) - C_1(b, a) = \{a, b\}. \tag{4.9}$$

Proposition 4.17. *Let $\mathbf{X}_1, \mathbf{X}_2$ be $(\star', \star)$-bimodule deformations of $\mathrm{X}_1, \mathrm{X}_2 \in \underline{\mathrm{Pic}}_{\mathcal{A}}(\mathcal{A})$, respectively. Let $D_1 = S(\mathbf{X}_1)$ and $D_2 = S(\mathbf{X}_2)$ be the induced contravariant connections on X_1 and X_2. If*

$$T = \sum_{r=0}^{\infty} \lambda^r T_r : \mathbf{X}_1 \longrightarrow \mathbf{X}_2$$

is a bimodule isomorphism, then $T_0 : \mathrm{X}_1 \longrightarrow \mathrm{X}_2$ *is an isomorphism preserving contravariant connections. In particular, if* T *is an equivalence,* $D_1 = D_2$.

Proof. Let us denote the left and right actions on $\mathbf{X}_i$ by

$$\bullet_i' = \sum_{r=0}^{\infty} \lambda^r R_r^{i'} \quad \text{and} \quad \bullet_i = \sum_{r=0}^{\infty} \lambda^r R_r^{i}, \quad i = 1, 2.$$

From $T(a \bullet_1' x) = a \bullet_2' T(x)$ and $T(x \bullet_1 a) = T(x) \bullet_2 a$, we get, in order λ,

$$\begin{aligned} T_1(ax) + T_0(R_1^{1'}(x, a)) &= R_1^{2'}(a, T_0(x)) + aT_1(x), \\ T_1(xa) + T_0(R_1^{1}(x, a)) &= R_1^{2}(T_0(x), a) + T_1(x)a. \end{aligned}$$

Subtracting the two equations and recalling that we are assuming that the classical left and right actions are the same, we get

$$T_0(D_1(a, x)) = T_0(R_1^1(x, a) - R_1^{1'}(x, a)) = R_1^2(T_0(x), a) - R_1^{2'}(a, T_0(x)) = D_2(a, T_0(x)).$$

□

As a result of Proposition 4.17, the semiclassical limit map S (4.8) induces a map of isomorphism classes

$$\widetilde{S} : \mathrm{Def}(\mathrm{X}, \star', \star) \longrightarrow \mathrm{Conn}(\mathrm{X}, \tau). \tag{4.10}$$

Remark 4.18. Let $\star$ be a deformation of $\mathcal{A}$ and $\mathrm{X} \in \underline{\mathrm{Pic}}_{\mathcal{A}}(\mathcal{A})$. Let D be a contravariant connection on X. A natural question is whether one can find $\star' \in \Phi_{\mathrm{X}}([\star])$, with $C_1' = C_1$, and a $(\star', \star)$-bimodule deformation of X with D as the semiclassical limit. For classes of deformations for which Hochschild 2-cochains are cohomologous to their skew-symmetric parts (e.g, for star products), one can always find $\star' \in \Phi_{\mathrm{X}}([\star])$ with $C_1' = C_1$ since Φ preserves Poisson brackets. In this case, a simple computation shows that one can choose $\star'$ so that there exists a $(\star', \star)$-bimodule deformation $\mathbf{X}$ of X with $S(\mathbf{X}) = D$, see [5].

4.3. **Deformation quantization of bimodules.** Let $\mathrm{X} \in \underline{\mathrm{Pic}}_{\mathcal{A}}(\mathcal{A})$, $\star$ be a deformation of $(\mathcal{A}, \{\ ,\ \})$ and $\star' \in \Phi_{\mathrm{X}}([\star])$. Following the notation in (4.3), we assume that $C_1 = C_1'$, and consider the semi-classical limit map

$$S : \underline{\mathrm{Def}}(\mathrm{X}, \star', \star) \longrightarrow \underline{\mathrm{Conn}}(\mathrm{X}, \tau)$$

as in (4.8).

Definition 4.19. A contravariant connection D on X is called *quantizable* if D is in the image of S.

Analogously to deformation of algebras, we call an element $\mathbf{X} \in S^{-1}(D)$ a $(\star', \star)$-bimodule deformation of X *in the direction of* D.

In this section, we will address the following two deformation quantization problems:

- First, whether any contravariant connection on X with curvature τ (determined by $\star'$ and $\star$) is quantizable (i.e, whether S is onto);

$*$ Second, how to classify equivalence classes of $(\star', \star)$-bimodule deformations of X in the direction of a fixed $D \in \operatorname{im}(S)$, i.e., equivalence classes of elements in $S^{-1}(D)$.

Let $\star$ be a deformation of $\mathcal{A}$, and consider the map

$$s : \operatorname{Equiv}(\boldsymbol{\mathcal{A}}) \longrightarrow \operatorname{PDer}(\mathcal{A}), \quad \operatorname{Id} + \lambda T_1 + O(\lambda^2) \mapsto T_1, \tag{4.11}$$

which clearly is a group homomorphism. Let $u = u_0 + O(\lambda) \in \boldsymbol{\mathcal{A}}$ be invertible. A simple computation shows that $s(\operatorname{Ad}(u)) = \{u_0, \cdot\}/u_0$. Since any invertible $u_0 \in \mathcal{A}$ is also invertible in $\boldsymbol{\mathcal{A}}$, it follows that

$$s(\operatorname{InnAut}(\boldsymbol{\mathcal{A}})) = \operatorname{IDer}(\mathcal{A}). \tag{4.12}$$

Definition 4.20. A Poisson derivation $\alpha \in \operatorname{PDer}(\mathcal{A})$ is called *quantizable* if there exists a $\star$-derivation σ with $\sigma = \alpha + O(\lambda)$.

Remark 4.21. Recall that any $\star$-self-equivalence T is of the form $\mathrm{e}^{\lambda\sigma}$ for a unique $\star$-derivation σ (see e.g. [8, Lem. 5]). So $\alpha \in \operatorname{PDer}(\mathcal{A})$ is quantizable if and only if it lies in the image of s. By (4.12), any integral derivation of $\mathcal{A}$ is quantizable.

Theorem 4.22. *Let* $\mathrm{X} \in \underline{\operatorname{Pic}}_{\mathcal{A}}(\mathcal{A})$, $\star$ *be a deformation of* $(\mathcal{A}, \{\,,\,\})$ *and* $\star' \in \Phi_{\mathrm{X}}([\star])$. *With the notation of (4.3), let us assume that* $C_1 = C_1'$. *The semiclassical limit map* $S : \underline{\operatorname{Def}}(\mathrm{X}, \star', \star) \longrightarrow \underline{\operatorname{Conn}}(\mathrm{X}, \tau)$ *is onto if and only if* $s : \operatorname{Equiv}(\boldsymbol{\mathcal{A}}') \longrightarrow \operatorname{PDer}(\mathcal{A})$ *is onto.*

Proof. For $\mathbf{X} \in \underline{\operatorname{Def}}(\mathrm{X}, \star', \star)$, let $D = S(\mathbf{X})$. For each $T = \operatorname{Id} + \lambda T_1 + O(\lambda) \in \operatorname{Equiv}(\boldsymbol{\mathcal{A}}')$, a simple computation shows that $S({}_T\mathbf{X}) = D - T_1$. This shows that if any Poisson derivation is quantizable, we can choose T conveniently and any contravariant connection on X with curvature τ is quantizable.

Now suppose S is onto. Let α be a Poisson derivation. Then there exists a deformation $\mathbf{Y}$ of X so that $S(\mathbf{Y}) = D + \alpha$. On the other hand, $\mathbf{Y} \cong {}_T\mathbf{X}$ for some $T \in \operatorname{Equiv}(\boldsymbol{\mathcal{A}}')$. Since $S({}_T\mathbf{X}) = D - T_1$, it follows from Proposition 4.17 that $D + \alpha$ and $D - T_1$ are isomorphic connections on X. By Lemma 4.8, this is the case if and only if

$$\alpha = -T_1 + u^{-1}\{\cdot, u\},$$

for some invertible $u \in \mathcal{A}$. But since both T_1 and $u^{-1}\{\cdot, u\}$ are quantizable, so is α. □

Keeping the notation of Theorem 4.22, we have the following result concerning classification of bimodule deformations.

Theorem 4.23. *Let* $D \in S(\underline{\operatorname{Def}}(\mathrm{X}, \star', \star))$. *The set of equivalence classes of bimodule deformations in* $S^{-1}(\mathrm{X}, D)$ *is in one-to-one correspondence with the set*

$$\frac{\{T \in \operatorname{Equiv}(\boldsymbol{\mathcal{A}}') \mid T_1 = 0\}}{\{\operatorname{Ad}(u) \in \operatorname{InnEquiv}(\boldsymbol{\mathcal{A}}') \mid u = 1 + O(\lambda) \in \boldsymbol{\mathcal{A}}'\}}. \tag{4.13}$$

Proof. If $\mathbf{X}$ is a deformation of (X, D), then, for $T = \operatorname{Id} + \lambda T_1 + O(\lambda^2) \in \operatorname{Equiv}(\boldsymbol{\mathcal{A}}')$, ${}_T\mathbf{X}$ deforms $(\mathrm{X}, D - T_1)$. So ${}_T\mathbf{X}$ deforms (X, D) if and only if $T_1 = 0$.

The bimodules $\mathbf{X}$ and ${}_T\mathbf{X}$ are isomorphic if and only if $T = \mathrm{Ad}(u)$, for some invertible $u = u_0 + O(\lambda) \in \boldsymbol{\mathcal{A}}'$. The induced bimodule isomorphism $\mathbf{X} \longrightarrow {}_T\mathbf{X}$ is given by

$$x \mapsto u \bullet' x, \quad x \in \mathbf{X}.$$

So this isomorphism is an equivalence if and only if $u_0 = 1$. □

5. Inner automorphisms of star-product algebras

In this section we briefly discuss the structure of the automorphism group of a star product $\star$ on a Poisson manifold (M, π). As usual, we let $\mathcal{A} = C^\infty(M)$ and $\boldsymbol{\mathcal{A}} = (C^\infty(M)[[\lambda]], \star)$.

5.1. **The star exponential.** We shall need the star exponential [2] as a technical tool and will recall its main properties below.

For a given $H \in \boldsymbol{\mathcal{A}}$ the *star exponential* $\mathrm{Exp}(H)$ is defined as the $t = 1$ value of the unique solution to the differential equation

$$\frac{d}{dt}\mathrm{Exp}(tH) = H \star \mathrm{Exp}(tH) \tag{5.1}$$

with initial condition $\mathrm{Exp}(0) = 1$. It is well-known that (5.1) has indeed a solution with the following properties, see e.g. [8, App. A].

Lemma 5.1. *Let $H = \sum_{r=0}^\infty \lambda^r H_r \in C^\infty(M)[[\lambda]]$. Then (5.1) has a unique solution $\mathrm{Exp}(tH)$ of the form*

$$\mathrm{Exp}(H) = \sum_{r=0}^\infty \lambda^r \mathrm{Exp}(H)_r \quad \textit{with} \quad \mathrm{Exp}(H)_r \in C^\infty(M), \tag{5.2}$$

where $\mathrm{Exp}(H)_0 = \mathrm{e}^{H_0}$ and $\mathrm{Exp}(H)_{r+1} = \mathrm{e}^{H_0}H_{r+1} + E_r$ with E_r depending on $H_0, \ldots, H_r$ only. The following additional properties hold.

1. *$H \star \mathrm{Exp}(H) = \mathrm{Exp}(H) \star H$ and $\mathrm{Exp}(tH) \star \mathrm{Exp}(sH) = \mathrm{Exp}((t+s)H)$.*
2. *Let M be connected. Then $\mathrm{Exp}(H) = 1$ if and only if H is constant and equal to $2\pi i m$, for some $m \in \mathbb{Z}$.*
3. *For all $f \in C^\infty(M)[[\lambda]]$ one has $\mathrm{Exp}(H) \star f \star \mathrm{Exp}(-H) = \mathrm{e}^{\mathrm{ad}(H)}(f)$, where $\mathrm{ad}(H)f = [H, f] = H \star f - f \star H$ is the $\star$-commutator.*
4. *$[f, g] = 0$ if and only if $[\mathrm{Exp}(f), \mathrm{Exp}(g)] = 0$ if and only if $[f, \mathrm{Exp}(g)] = 0$. In this case, $\mathrm{Exp}(f) \star \mathrm{Exp}(g) = \mathrm{Exp}(f + g)$.*

Lemma 5.2. *Let $\mathcal{O} \subseteq M$ be an open contractible subset and let $U \in C^\infty(\mathcal{O})[[\lambda]]$ be invertible. Then there exists a function $H \in C^\infty(\mathcal{O})[[\lambda]]$ such that $\mathrm{Exp}(H) = U$ and any two such functions differ by a constant $2\pi i m$, with $m \in \mathbb{Z}$.*

5.2. **Locally inner derivations.** Let $\sigma : \boldsymbol{\mathcal{A}} \to \boldsymbol{\mathcal{A}}$ be a $\mathbb{C}[[\lambda]]$-linear derivation of $\star$. We call σ *locally inner* if, on each contractible open subset $\mathcal{O} \subseteq M$, there exists a function $H \in C^\infty(\mathcal{O})[[\lambda]]$ such that

$$\sigma\big|_{C^\infty(\mathcal{O})[[\lambda]]} = \mathrm{ad}(H). \tag{5.3}$$

The $\mathbb{C}[[\lambda]]$-module of locally inner derivations is denoted by $\mathrm{LocInnDer}(\boldsymbol{\mathcal{A}})$.

Definition 5.3. Let $A \in Z^1(M, \mathbb{C})[[\lambda]]$ be a closed one-form. We define a global $\star$-derivation δ_A by

$$\delta_A\big|_{C^\infty(\mathcal{O})[[\lambda]]} = \mathrm{ad}(H), \tag{5.4}$$

where $\mathcal{O}$ is a contractible open subset of M and H is a local function with $dH = A$ on $\mathcal{O}$.

It is clear that δ_A is well-defined. It is a locally inner derivation by construction and gives rise to a $\mathbb{C}[[\lambda]]$-linear map

$$\delta : Z^1(M, \mathbb{C})[[\lambda]] \longrightarrow \mathrm{LocInnDer}(\boldsymbol{\mathcal{A}}), \quad A \mapsto \delta_A. \tag{5.5}$$

In the symplectic case, the map δ is known to be a bijection between closed one-forms and the star product derivations, see e.g. [18]. In the Poisson case, δ is generally neither injective nor surjective. Moreover, the (non-)bijectivity of δ depends on $\star$ itself and not only on the Poisson bracket as the following example illustrates.

Example 5.4. *Consider a star product $\star'$ on a symplectic manifold, and let $\star$ be defined by replacing λ by λ^2 in $\star'$. Then $\star$ is a deformation of the zero Poisson bracket and δ is still bijective. Note that, in this case, the Poisson center and the center of $\star$ do not coincide. On the other hand, for the trivial star product corresponding to the zero Poisson bracket, any vector field is a derivation but δ_A is identically zero.*

The situation becomes easier when $\star$ has trivial center.

Lemma 5.5. *Suppose the center of $(C^\infty(M)[[\lambda]], \star)$ is trivial. Then δ is injective and surjective onto the set of locally inner derivations.*

Proof. Let $\{\mathcal{O}_\alpha\}$ be a cover of M by contractible open sets. Given a locally inner derivation σ with $\sigma = \mathrm{ad}(H_\alpha)$ on $\mathcal{O}_\alpha$ and $\sigma = \mathrm{ad}(H_\beta)$ on $\mathcal{O}_\beta$, we see that $H_\alpha - H_\beta$ is a central function on $\mathcal{O}_\alpha \cap \mathcal{O}_\beta$ hence locally constant. Thus $A = dH_\alpha$ defines a *global* one-form and $\sigma = \delta_A$. On the other hand, if $\delta_A = 0$ then $\mathrm{ad}(H) = 0$. So H is constant and $0 = dH = A$. □

We finally observe the following simple result.

Lemma 5.6. *Let $A \in Z^1(M, \mathbb{C})[[\lambda]]$, and let σ be an arbitrary $\star$-derivation. Then $[\sigma, \delta_A]$ is inner.*

Proof. Obvious, as $\sigma H_\alpha = \sigma H_\beta$ for $A = dH_\alpha = dH_\beta$ on $\mathcal{O}_\alpha \cap \mathcal{O}_\beta$. □

5.3. **Inner automorphisms of star products.** We will now characterize the inner automorphism of the star-product algebra $\boldsymbol{\mathcal{A}} = (C^\infty(M)[[\lambda]], \star)$.

Theorem 5.7.
(1) *Let $u = \sum_{r=0}^\infty \lambda^r u_r \in \boldsymbol{\mathcal{A}}$ be invertible and let $T = \mathrm{Ad}(u)$. Then there exists a closed one-form $A \in Z^1(M, \mathbb{C})[[\lambda]]$ with $T = \mathrm{e}^{\delta_A}$. Moreover, A_0 is $2\pi i$-integral and the A_r are exact for $r \geq 1$.*
(2) *The derivations $\mathrm{LogInnAut}(\boldsymbol{\mathcal{A}})$ of $\boldsymbol{\mathcal{A}}$ which exponentiate to inner automorphisms form an abelian group under addition and we have the following inclusions of groups*

$$\mathrm{InnDer}(\boldsymbol{\mathcal{A}}) \subseteq \mathrm{LogInnAut}(\boldsymbol{\mathcal{A}}) \subseteq \delta(Z^1(M, \mathbb{C})[[\lambda]]) \subseteq \mathrm{LocInnDer}(\boldsymbol{\mathcal{A}}). \tag{5.6}$$

(3) *The map*

$$2\pi i H^1_{\mathrm{dR}}(M,\mathbb{Z}) \ni [A] \mapsto [\delta_A] \in \frac{\mathrm{LogInnAut}(\boldsymbol{\mathcal{A}})}{\mathrm{InnDer}(\boldsymbol{\mathcal{A}})} \tag{5.7}$$

is a surjective group homomorphism. If the center of $\star$ is trivial, it is a bijection.

Proof. For the first part, recall that $T = \mathrm{e}^{\lambda\sigma}$ with a unique $\star$-derivation σ, see Remark 4.21. Let $\{\mathcal{O}_\alpha\}$ be a cover of M by contractible open sets. By Lemma 5.2, there exist local functions $H_\alpha \in C^\infty(\mathcal{O}_\alpha)[[\lambda]]$ with $u = \mathrm{Exp}(H_\alpha)$ on $\mathcal{O}_\alpha$. Hence $T = \mathrm{e}^{\mathrm{ad}(H_\alpha)}$ on $\mathcal{O}_\alpha$. Since the functions H_α are unique up to constants, the one-form $A = dH_\alpha$ is well-defined globally. Thus $T = \mathrm{e}^{\delta_A}$. Finally, the H_α differ by constants in $2\pi i\mathbb{Z}$ whence A_0 is $2\pi i$-integral, and the A_r are exact for $r \geq 1$. The second part follows from the first part. The third part is a consequence of (5.6) and Lemma 5.5. □

Let $\mathcal{Z}_\pi(\mathcal{A})$ denote the Poisson center of $(\mathcal{A}, \{\ ,\ \})$, and let $\mathcal{Z}(\boldsymbol{\mathcal{A}})$ denote the $\star$-center of $\boldsymbol{\mathcal{A}}$. Then we define the following property of a star product $\star$:

C1) There exists a linear map $\rho_0 : \mathcal{Z}_\pi(\mathcal{A}) \longrightarrow \mathcal{Z}(\boldsymbol{\mathcal{A}})$ with $\rho_0(f) = f + O(\lambda)$.

Note that, in this case, ρ_0 actually extends to a $\mathbb{C}[[\lambda]]$-linear bijection between $\mathcal{Z}_\pi(\mathcal{A})[[\lambda]]$ and the center $\mathcal{Z}(\boldsymbol{\mathcal{A}})$. Hence the center $\mathcal{Z}(\boldsymbol{\mathcal{A}})$ is a deformation of the sub-space $\mathcal{Z}_\pi(\mathcal{A})$ in the sense of [4, Def. 30]. In Section 6 we will discuss examples of star products for which such a ρ_0 exists.

Theorem 5.8. *Let $T = \mathrm{Ad}(u)$ be an inner automorphism of $\boldsymbol{\mathcal{A}}$ of the form $T = \mathrm{Id} + O(\lambda^2)$ and suppose that $\star$ satisfies C1). Then there exists $u' = 1 + O(\lambda)$ such that $T = \mathrm{Ad}(u') = \mathrm{e}^{\mathrm{ad}(H)}$ for some global function $H \in \lambda C^\infty(M)[[\lambda]]$.*

Proof. Let us write $T(f) = u \star f \star v$. Evaluating $u \star f \star v$ in first order, which has to vanish according to our assumptions, we get the condition

$$u_0 f v_1 + u_1 f v_0 + C_1(u_0, f) v_0 + u_0 C_1(f, v_0) = 0.$$

Since T^{-1} is also of the form $\mathrm{Id} + O(\lambda^2)$, we obtain the same equation with the roles of u and v exchanged. Taking their difference, we get

$$v_0\{u_0, f\} = -u_0\{f, v_0\}.$$

Using $u_0 v_0 = 1$, we conclude that

$$\{u_0, f\} = 0 \ \text{ for all } \ f \in C^\infty(M),$$

and therefore $u_0 \in \mathcal{Z}_\pi(\mathcal{A})$. So $\rho_0(u_0) = u_0 + O(\lambda) \in \mathcal{Z}(\boldsymbol{\mathcal{A}})$. Since u_0 is invertible, so is $\rho_0(u_0)$, and we can define

$$u' = \rho_0(u_0)^{-1} \star u.$$

Note that $v' = \rho_0(u_0) \star v$ is the $\star$-inverse of u', and clearly $T = \mathrm{Ad}(u) = \mathrm{Ad}(u')$ as $\rho_0(u_0)$ is central. Since in this case the formal series of the star logarithm trivially converges in the λ-adic topology, we can find $H \in \lambda C^\infty(M)[[\lambda]]$ with $u' = \mathrm{Exp}(H)$. Hence $T(f) = \mathrm{Exp}(H) \star f \star \mathrm{Exp}(-H) = \mathrm{e}^{\mathrm{ad}(H)}$. □

Corollary 5.9. *If $\star$ satisfies C1) then there exists a bijection*

$$\frac{\{T \in \mathrm{Equiv}(\boldsymbol{\mathcal{A}}) \mid T_1 = 0\}}{\{\mathrm{Ad}(u) \in \mathrm{InnEquiv}(\boldsymbol{\mathcal{A}}) \mid u = 1 + O(\lambda)\}} \longrightarrow \mathrm{OutDer}(\boldsymbol{\mathcal{A}}), \tag{5.8}$$

where $\mathrm{OutDer}(\boldsymbol{\mathcal{A}})$ *denotes the outer derivations of* $\boldsymbol{\mathcal{A}}$.

Proof. There is a one-to-one correspondence between self-equivalences of $\boldsymbol{\mathcal{A}}$ and $\star$-derivations given by

$$T = \mathrm{e}^{\lambda\sigma} \mapsto \sigma.$$

Under this bijection, the set $\{T \in \mathrm{Equiv}(\boldsymbol{\mathcal{A}}),\ T_1 = 0\}$ is mapped bijectively onto derivations of the form $\lambda\sigma$, and, by Theorem 5.8, the set $\{\mathrm{Ad}(u) \in \mathrm{InnEquiv}(\boldsymbol{\mathcal{A}}),\ u = 1 + O(\lambda)\}$ corresponds to inner derivations of the form $\lambda\sigma$. So the result follows. □

As the lefthand side of (5.8) is a group, one may ask which group structure on $\mathrm{OutDer}(\boldsymbol{\mathcal{A}})$ is obtained from this bijection. This is illustrated in the following example.

Example 5.10. *Let $\star$ be the trivial deformation, i.e. the undeformed product. Then there are no non-trivial inner automorphisms and the equivalence transformations with $T_1 = 0$ are of the form $\mathrm{e}^{\lambda\mathcal{L}_X}$ with some vector field $X \in \lambda\Gamma^\infty(TM)[[\lambda]]$. Thus $\mathrm{e}^{\lambda\mathcal{L}_X}\mathrm{e}^{\lambda\mathcal{L}_Y} = \mathrm{e}^{\lambda\mathcal{L}_{H(X,Y)}}$, where*

$$H(X,Y) = X + Y + \frac{1}{2}[X,Y] + \cdots$$

is the Baker-Campbell-Hausdorff series, which is well-defined as X, Y start in order λ. Thus the outer derivations inherit the non-abelian group structure

$$X \cdot Y = \frac{1}{\lambda} H(\lambda X, \lambda Y).$$

6. Bimodule deformations of line bundles over Poisson manifolds

We will now specialize the discussion of deformation quantization of invertible bimodules to the context of line bundles over a Poisson manifold (M, π). We will consider star products $\star$ on M satisfying property C1) and the following analogous property for derivations:

C2) There exists a linear map $\rho_1 : \mathrm{PDer}(\mathcal{A}) \longrightarrow \mathrm{Der}(\boldsymbol{\mathcal{A}})$, with $\rho_1(X) = \mathcal{L}_X + O(\lambda)$, such that, for Hamiltonian vector fields, $\rho_1(X_H) = \frac{i}{\lambda}\mathrm{ad}(H)$.

Here $\mathcal{L}_X$ is the Lie derivative of X. The map ρ_1 extends to a $\mathbb{C}[[\lambda]]$-linear bijection between $\mathrm{PDer}(\mathcal{A})[[\lambda]]$ and $\mathrm{Der}(\boldsymbol{\mathcal{A}})$ yielding a deformation of the subspace $\mathrm{PDer}(\mathcal{A})$.

We call $\star$-derivations of the form $\frac{i}{\lambda}\mathrm{ad}(H)$ *quasi-inner*. Since we assume that ρ_1 maps Hamiltonian vector fields into quasi-inner $\star$-derivations, a simple induction shows the following lemma.

Lemma 6.1. *Let $\star$ be a star product on (M, π) satisfying C2). Then outer $\star$-derivations are in one-to-one correspondence with $H^1_\pi(M, \mathbb{C})[[\lambda]]$.*

Before we move on, let us remark that one can always find star products on (M, π) satisfying conditions C1) and C2).

Example 6.2. *Let (M,ω) be a symplectic manifold, and let $\star$ be any star product on M. In this case, the Poisson center of $\mathcal{A}$ is trivial and one can take ρ_0 to be the trivial embedding. As mentioned in Section 5.2, the map δ in (5.5) is a bijection in the symplectic case. The correspondence between symplectic vector fields and closed one forms,*

$$X \mapsto \mathrm{i}_X\omega = \omega(X,\,\cdot\,),$$

induces a one-to-one correspondence

$$\hat{\delta} : \chi_s(M)[[\lambda]] \longrightarrow \mathrm{Der}(\boldsymbol{\mathcal{A}}),\ \ X \mapsto \frac{i}{\lambda}\delta_A, \quad \text{with} \quad A = \mathrm{i}_X\omega,$$

where $\chi_s(M)$ denotes the space of symplectic vector fields. Using the convention in (3.3), a simple computation shows that

$$\hat{\delta}_X(f) = \mathcal{L}_X + O(\lambda).$$

It is easy to check that $\hat{\delta}$ maps Hamiltonian vector fields on M onto quasi-inner $\star$-derivations. So one can take $\hat{\delta}$ as ρ_1. As a consequence, we have a group isomorphism between outer $\star$-derivations and elements in $H^1_{\mathrm{dR}}(M,\mathbb{C})[[\lambda]]$, see e.g. [18].

In order to describe the group structure on $H^1_{\mathrm{dR}}(M,\mathbb{C})[[\lambda]]$ induced by (5.8), let A, A' be closed one-forms. From Lemma 5.6 and the Baker-Campbell-Hausdorff theorem, it follows that $\mathrm{e}^{\delta_A}\mathrm{e}^{\delta_{A'}} = \mathrm{e}^{\delta_{A+A'}}$ modulo some inner automorphism. Thus the induced group structure on $\mathrm{OutDer}(\boldsymbol{\mathcal{A}})$ is just the abelian one, quite in contrast with Example 5.10.

Example 6.3. *Let (M,π) be an arbitrary Poisson manifold, and let $\mathcal{U}$ be a formality on M* [22], *i.e., an L_∞-quasi-isomorphism from the differential graded Lie algebra of multivector fields on M (with zero differential and Schouten bracket) into the differential graded Lie algebra of multidifferential operators on M (with Hochschild differential and Gerstenhaber bracket). It is well-known that $\mathcal{U}$ is completely determined by its Taylor coefficients $\mathcal{U}_p$, $p \geq 1$. Kontsevich showed in* [22] *that formalities exist on arbitrary Poisson manifolds and that one can take $\mathcal{U}_1$ to be the natural embedding of vector fields into differential operators. Given $\mathcal{U}$, one can define a star product on (M,π) by*

$$f \star g := fg + \sum_{r=1}^{\infty} \frac{\lambda^r}{r!}\mathcal{U}_r(\pi,\dots,\pi)(f\otimes g), \quad f,g \in C^\infty(M). \tag{6.1}$$

We call a star product associated to a formality as in (6.1) a Kontsevich's star product. *More generally, Konstevich has shown that any star product on (M,π) is equivalent to one obtained as in (6.1) with π replaced by a formal Poisson structure $\pi_\lambda = \pi + O(\lambda)$.*

As a consequence of the formality equations, one can show that Kontsevich's star products satisfy C1) and C2) (see e.g. [10, 11, 15, 21]*) for*

$$\rho_0(f) = f + \sum_{r=1}^{\infty}\frac{\lambda^r}{r!}\mathcal{U}_{r+1}(f,\pi,\dots,\pi), \quad \text{and} \quad \rho_1(X) = \mathcal{L}_X + \sum_{r=1}^{\infty}\frac{\lambda^r}{r!}\mathcal{U}_{r+1}(X,\pi,\dots,\pi).$$

Example 6.4. *A more geometric way of obtaining global star products on (M,π) is described in [10, 11], where the authors use the formality on $\mathbb{R}^d$ combined with a Fedosov-type procedure [13]. It is shown in [11] that these star products can also be constructed satisfying C1) and C2). We will refer to these star products as* CFT star products.

Theorem 6.5. *Let $\star$ be a star product on (M,π) satisfying C1) and C2) (in particular, symplectic, CFT and Kontsevich's star products). Let $L \to M$ be a complex line bundle, and let $\star' \in \Phi_L([\star])$. Let D be a contravariant connection on L with curvature $-\tau$ determined by $\star$ and $\star'$. Then there exists a $(\star',\star)$-bimodule deformation $\mathbf{X}$ of $\mathrm{X} = \Gamma^\infty(L)$, with $S(\mathbf{X}) = D$. Moreover, equivalence classes of bimodule deformations in $S^{-1}(D)$ are in one-to-one correspondence with $H^1_\pi(M,\mathbb{C})[[\lambda]]$.*

Proof. For the existence part, note that property C1) guarantees that the map s (4.11) is onto. So the result follows from Theorem 4.22. As for the second assertion, note that, since $\star$ satisfies C1), Corollary 5.9 holds. Since property C2) is satisfied, we can apply Lemma 6.1, and the result follows. □

7. The map cl_* for star-product algebras

7.1. **The kernel of cl_*.** Let (M,π) be a Poisson manifold and $\star$ be a star product on M satisfying C1) and C2). As usual, we let $\mathcal{A} = C^\infty(M)$ and $\boldsymbol{\mathcal{A}} = (C^\infty(M)[[\lambda]], \star)$. Using ρ_1, we consider the map

$$\mathrm{Equiv}(\boldsymbol{\mathcal{A}}) \longrightarrow \mathrm{PDer}(\mathcal{A}) \times \{T \in \mathrm{Equiv}(\boldsymbol{\mathcal{A}}),\ T_1 = 0\}, \quad T \mapsto (T_1, \mathrm{e}^{-\lambda\rho_1(T_1)} \circ T), \tag{7.1}$$

which is easily seen to be a bijection. In the symplectic case, for $\rho_1 = \hat{\delta}$, the map (7.1) restricts to a bijection

$$\mathrm{InnEquiv}(\mathcal{A}) \longrightarrow \mathrm{IDer}(A) \times \{T \in \mathrm{InnEquiv}(\boldsymbol{\mathcal{A}}),\ T_1 = 0\}, \tag{7.2}$$

which does not seem to be the case in general. However, there always exists a bijection (7.2) given by

$$T = \mathrm{Ad}(u) \mapsto (\{u_0, \cdot\}/u_0, \mathrm{e}^{-\lambda \mathrm{Ad}(u_0)} \circ T). \tag{7.3}$$

By Theorem 5.8, (7.3) actually establishes a bijection

$$\mathrm{InnEquiv}(\mathcal{A}) \longrightarrow \mathrm{IDer}(A) \times \{T = \mathrm{Ad}(u) \in \mathrm{InnEquiv}(\boldsymbol{\mathcal{A}}),\ u = 1 + O(\lambda)\}. \tag{7.4}$$

Note that (7.1) and (7.2) imply (generally, in a noncanonical way) the existence of a a bijection

$$\mathrm{OutEquiv}(\boldsymbol{\mathcal{A}}) \longrightarrow \frac{\mathrm{PDer}(\mathcal{A})}{\mathrm{IDer}(\mathcal{A})} \times \frac{\{T \in \mathrm{Equiv}(\boldsymbol{\mathcal{A}}),\ T_1 = 0\}}{\{T = \mathrm{Ad}(u),\ u_0 = 1\}} \cong \frac{H^1_\pi(M,\mathbb{C})}{2\pi i H^1_\pi(M,\mathbb{Z})} + \lambda H^1_\pi(M,\mathbb{C})[[\lambda]],$$

where the last identification follows from (4.2), Corollary 5.9 and Lemma 6.1. As a consequence of Corollary 3.11, we can state the following theorem.

Theorem 7.1. *For a star product $\star$ satisfying properties C1) and C2) (e.g., symplectic, CFT and Kontsevich's or star products), there is a one-to-one correspondence*

$$\ker(\mathrm{cl}_*) \xrightarrow{\sim} \frac{H^1_\pi(M,\mathbb{C})}{2\pi i H^1_\pi(M,\mathbb{Z})} + \lambda H^1_\pi(M,\mathbb{C})[[\lambda]]. \tag{7.5}$$

In particular, cl_* *is injective if and only if* $H^1_\pi(M,\mathbb{C})=0$.

We remark that the identification (7.5) is a canonical group isomorphism for symplectic star products. In this case, cl_* is injective if and only if $H^1_{\mathrm{dR}}(M,\mathbb{C})=0$.

However, in general (7.5) is not a group morphism for the canonical abelian group structure on the right-hand side: If we consider Example 5.10, we see that the group structure on the left-hand side of (7.5) is non-abelian.

7.2. **The image of cl_* for symplectic star products.** Let (M,ω) be a symplectic manifold. As shown in [3, 12, 27, 32], there exists a bijection

$$c:\mathrm{Def}(M,\omega)\longrightarrow \frac{1}{i\lambda}[\omega]+H^2_{\mathrm{dR}}(M)[[\lambda]], \tag{7.6}$$

characterizing the moduli space $\mathrm{Def}(M,\omega)$ in terms of the de Rham cohomology of M. For a star product $\star$ on M, $c(\star)$ is called its *characteristic class.*

Let $\star$ be a star product on M with characteristic class

$$c(\star)=\frac{1}{i\lambda}[\omega]+\sum_{r=0}^{\infty}\lambda^r[\omega_r].$$

It is well known that the action of the group of symplectomorphisms of M on star products, in terms of characteristic classes, is given by [18]

$$c(\psi^*(\star))=\psi^*c(\star)=\frac{1}{i\lambda}[\omega]+\sum_{r=0}^{\infty}\lambda^r[\psi^*\omega_r]. \tag{7.7}$$

The simple description of this action is the main reason why we restrict ourselves to symplectic star products in this section.

The main result in [8] asserts that, if $L\to M$ is a complex line bundle and $\star'\in\Phi_L([\star])$ (see Remark 3.17), then

$$c(\star')=c(\star)+2\pi i c_1(L), \tag{7.8}$$

where $c_1(L)\in H^2_{dR}(M,\mathbb{C})$ is the image under the canonical map

$$e:H^2(M,\mathbb{Z})\longrightarrow H^2_{\mathrm{dR}}(M,\mathbb{C}) \tag{7.9}$$

of the Chern class of L in $H^2(M,\mathbb{Z})$. Combining these results with Corollary 3.18, we obtain the following:

Corollary 7.2. *Let $\star$ be a star product on (M,ω), with $c(\star)=(1/i\lambda)[\omega]+\sum_{r=0}^{\infty}\lambda^r[\omega_r]$. Then*

$$\mathrm{im}(\mathrm{cl}^r_*)=\{l\in H^2(M,\mathbb{Z})\mid \exists\psi\in\mathrm{Sympl}(M)\ \textit{so that}\ e(l)=\psi^*[\omega_0]-[\omega_0]\ \textit{and}\ \psi^*[\omega_r]=[\omega_r],\ r\geq 0\}.$$

For $l\in\mathrm{im}(\mathrm{cl}^r_*)$, let

$$P_l=\{\psi\in\mathrm{Sympl}(M)\mid e(l)=\psi^*[\omega_0]-[\omega_0]\ \text{and}\ \psi^*[\omega_r]=[\omega_r],\ r\geq 0\}.$$

Recall from Example 2.6 that $\mathrm{Pic}(C^\infty(M))=\mathrm{Diff}(M)\ltimes\mathrm{Pic}(M)$, with action given by pull-back.

Corollary 7.3. *The image of cl_* in $\mathrm{Pic}(C^\infty(M))=\mathrm{Diff}(M)\ltimes\mathrm{Pic}(M)$ is given by*

$$\mathrm{im}(\mathrm{cl}_*)=\{(\psi,l)\in\mathrm{Diff}(M)\ltimes\mathrm{Pic}(M)\mid\psi\in P_l\quad\textit{and}\quad l\in\mathrm{im}(\mathrm{cl}^r_*)\}.$$

Let $\mathrm{Tor}(H^2(M,\mathbb{Z}))$ denote the subgroup of torsion elements of $H^2(M,\mathbb{Z})$. We recall that $\mathrm{Tor}(H^2(M,\mathbb{Z})) = \ker(e)$. This subgroup can also be described as isomorphism classes of line bundles over M admitting a flat connection. The following result is a consequence of the Corollaries 7.2 and 7.3.

Corollary 7.4. *Let $\star$ be a star product on M such that $[\psi^*(\star)] = [\star]$ for all $\psi \in \mathrm{Sympl}(M)$. Then*

$$\mathrm{im}(\mathrm{cl}^r_*) = \mathrm{Tor}(H^2(M,\mathbb{Z})) \quad \textit{and} \quad \mathrm{im}(\mathrm{cl}_*) = \mathrm{Sympl}(M) \ltimes \mathrm{Tor}(H^2(M,\mathbb{Z})),$$

with action given by pull-back. Moreover, by Theorem 6.5, if $H^1_{\mathrm{dR}}(M,\mathbb{C}) = 0$, then $\mathrm{Pic}(\boldsymbol{\mathcal{A}}) \cong \mathrm{Sympl}(M) \ltimes \mathrm{Tor}(H^2(M,\mathbb{Z}))$.

Let us discuss a few examples.

Example 7.5. *Let us consider $M = \mathbb{R}^{2n}$, equipped with its canonical symplectic form. In this case, $\mathrm{Pic}_{\mathcal{A}}(\mathcal{A}) \cong H^2(\mathbb{R}^{2n},\mathbb{Z}) = 0$ and*

$$\mathrm{Pic}(\mathcal{A}) \cong \mathrm{Diff}(\mathbb{R}^{2n}).$$

By Corollary 7.4, for any symplectic star product $\star$ on $\mathbb{R}^{2n}$,

$$\mathrm{Pic}(\boldsymbol{\mathcal{A}}) \cong \mathrm{Sympl}(\mathbb{R}^{2n}).$$

Example 7.6. *More generally, for any symplectic manifold (M,ω), we can always choose a star product $\star$ satisfying $[\psi^*(\star)] = [\star]$ for all $\psi \in \mathrm{Sympl}(M)$: for example, one can take $\star$ with the trivial class*

$$c(\star) = \frac{1}{i\lambda}[\omega].$$

In this case, $\mathrm{im}(\mathrm{cl}^r_*) = \mathrm{Tor}(H^2(M,\mathbb{Z}))$ and $\mathrm{im}(\mathrm{cl}_*) = \mathrm{Sympl}(M) \ltimes \mathrm{Tor}(H^2(M,\mathbb{Z}))$.

Example 7.7. *Suppose (M,ω) is a symplectic manifold such that $H^2_{\mathrm{dR}}(M,\mathbb{C})$ is one dimensional and $[\omega] \neq 0$. Then, for any $\psi \in \mathrm{Sympl}(M)$, the pull-back map*

$$\psi^* : H^2_{\mathrm{dR}}(M,\mathbb{C}) \longrightarrow H^2_{\mathrm{dR}}(M,\mathbb{C})$$

must be the identity. As a result of (7.7), for any star product $\star$,

$$\psi^*([\star]) = [\star] \quad \textit{for all} \quad \psi \in \mathrm{Sympl}(M).$$

Hence, by Corollary 7.2, $\mathrm{im}(\mathrm{cl}^r_*) = \mathrm{Tor}(H^2(M,\mathbb{Z}))$ and $\mathrm{im}(\mathrm{cl}_*) = \mathrm{Sympl}(M) \ltimes \mathrm{Tor}(H^2(M,\mathbb{Z}))$.

In particular, if M is a Riemann surface (equipped its area form) or $\mathbb{C}P^n$ (with the Fubini-Study symplectic form), then for any star product $\star$ on M we have

$$\mathrm{im}(\mathrm{cl}^r_*) = 0 \quad \textit{and} \quad \mathrm{im}(\mathrm{cl}_*) \cong \mathrm{Sympl}(M).$$

In the case $M = \mathbb{C}P^n$, since $H^1_{\mathrm{dR}}(\mathbb{C}P^n,\mathbb{C}) = 0$, we have

$$\mathrm{Pic}(\boldsymbol{\mathcal{A}}) \cong \mathrm{Sympl}(\mathbb{C}P^n)$$

for any star product.

Example 7.8. *In general, however,* $\mathrm{im}(\mathrm{cl}^r_*)$ *may contain non-torsion elements: Let* Σ *be a Riemann surface, and let* $M = T^*\Sigma$*, equipped with its canonical symplectic form. Recall that the natural projection* $\mathrm{pr} : M \longrightarrow \Sigma$ *induces an isomorphism*

$$\mathrm{pr}^* : H^2_{\mathrm{dR}}(\Sigma, \mathbb{C}) \longrightarrow H^2_{\mathrm{dR}}(M, \mathbb{C})$$

For $f \in \mathrm{Diff}(\Sigma)$*, let* $f^\sharp \in \mathrm{Sympl}(M)$ *be its cotangent lift. We have the following commutative diagram:*

$$\begin{array}{ccc} H^2_{\mathrm{dR}}(\Sigma, \mathbb{C}) & \xrightarrow{f^*} & H^2_{\mathrm{dR}}(\Sigma, \mathbb{C}) \\ {\scriptstyle \mathrm{pr}^*}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}^*} \\ H^2_{\mathrm{dR}}(M, \mathbb{C}) & \xrightarrow{(f^\sharp)^*} & H^2_{\mathrm{dR}}(M, \mathbb{C}). \end{array}$$

Let

$$\mathrm{Diff}(\Sigma)^* := \{f^* : H^2_{\mathrm{dR}}(\Sigma, \mathbb{C}) \longrightarrow H^2_{\mathrm{dR}}(\Sigma, \mathbb{C}) \ , \ f \in \mathrm{Diff}(\Sigma)\}.$$

Fixing an orientation in Σ*, we can identify* $\mathrm{Diff}(\Sigma)^*$ *with* $\mathbb{Z}_2 = \{\mathrm{Id}, -\mathrm{Id}\}$*. Therefore*

$$\mathrm{Sympl}(M)^* := \{\psi^* : H^2_{\mathrm{dR}}(M, \mathbb{C}) \longrightarrow H^2_{\mathrm{dR}}(M, \mathbb{C}) \ , \ \psi \in \mathrm{Sympl}(M)\} = \{\mathrm{Id}, -\mathrm{Id}\}.$$

Let us pick $[\omega_0] \in H^2_{\mathrm{dR}}(M, \mathbb{Z})$*, and let* $\star$ *be a star product on* M *with*

$$c(\star) = \frac{1}{i\lambda}[\omega] + [\omega_0] = [\omega_0],$$

since $[\omega] = 0$*. Then, by Corollary 7.2,*

$$\mathrm{im}(\mathrm{cl}^r_*) = \{0, -2[\omega_0]\}.$$

From Corollary 7.3, it follows that

$$\mathrm{im}(\mathrm{cl}_*) = A \cup B,$$

where

$$\begin{aligned} A &= \{(\psi, 0) \in \mathrm{Sympl}(M) \ltimes H^2(M, \mathbb{Z}) \,|\, \psi^* = \mathrm{Id} : H^2_{\mathrm{dR}}(M, \mathbb{C}) \longrightarrow H^2_{\mathrm{dR}}(M, \mathbb{C})\}, \\ B &= \{(\phi, -2[\omega_0]) \in \mathrm{Sympl}(M) \ltimes H^2(M, \mathbb{Z}) \,|\, \phi^* = -\mathrm{Id} : H^2_{\mathrm{dR}}(M, \mathbb{C}) \longrightarrow H^2_{\mathrm{dR}}(M, \mathbb{C})\}. \end{aligned}$$

It is simple to check that $A \cdot A \subseteq A$*,* $B \cdot A \subseteq B$*,* $A \cdot B \subseteq B$*, and* $B \cdot B \subseteq A$*, and this shows directly that* $\mathrm{im}(\mathrm{cl}_*)$ *is a subgroup of* $\mathrm{Pic}(\mathcal{A}) = \mathrm{Sympl}(M) \ltimes H^2(M, \mathbb{Z})$*.*

More complicated examples illustrating the nontriviality of $\mathrm{im}(\mathrm{cl}^r_*)$ are obtained, e.g., by considering $M = T^*(T^3)$ or $M = T^4$. It seems hard to describe exactly how big the image of cl^r_* can be in general. We have, however, the following result.

Theorem 7.9. *Let* $\star$ *be a star product on* (M, ω)*, and suppose* $H^2_{dR}(M, \mathbb{C})$ *is finite dimensional. Then the map* cl^r_* *is onto if and only if* $\mathrm{Pic}(M) = H^2(M, \mathbb{Z})$ *only contains torsion elements.*

Proof. It follows from Corollary 7.2 that $\mathrm{Tor}(H^2(M, \mathbb{Z})) \subseteq \mathrm{im}(\mathrm{cl}^r_*)$, and this shows the “if” direction.

For the converse, it suffices to show that

$$H^2(M, \mathbb{Z}) \nsubseteq \{l \in H^2(M, \mathbb{Z}) \mid \exists \psi \in \mathrm{Sympl}(M) \text{ so that } e(l) = \psi^*[\omega_0] - [\omega_0]\}.$$

Fixing a basis of $H^2_{\mathrm{dR}}(M,\mathbb{Z})$, we can identify $H^2_{\mathrm{dR}}(M,\mathbb{C})$ with $\mathbb{C}^m$, and $H^2_{\mathrm{dR}}(M,\mathbb{Z})$ with the lattice $\mathbb{Z}^m$. Let us assume $H^2(M,\mathbb{Z})$ has non-torsion elements, so that $m \geq 1$.

For $\psi \in \mathrm{Sympl}(M)$, ψ^* must preserve the integer lattice, and therefore $\mathrm{Sympl}(M)^* \subseteq \mathrm{GL}(m,\mathbb{Z})$. So it suffices to show that, for any fixed $v \in \mathbb{C}^m$,

$$\mathbb{Z}^m \nsubseteq \{Av - v\ ,\ \text{for } A \in \mathrm{GL}(m,\mathbb{Z})\}.$$

Suppose $v = (v_1,\ldots,v_m) \in \mathbb{C}^m$ is such that, for any $l \in \mathbb{Z}^m$, there exists $A \in \mathrm{GL}(m,\mathbb{Z})$ with $l = Av - v$. We will first show that, in this case, there exists a vector $r_0 \in \mathbb{Q}^m$ with the same property as v.

Let $E = \mathrm{span}_{\mathbb{Q}}\{v_1,\ldots,v_m\}$. It follows from our assumptions that $1 \in E$. So we can find a basis for the $\mathbb{Q}$-vector space E of the form $\{1, x_1, \ldots, x_s\}$. Using this basis, we write

$$v = r_0 + r_1 x_1 + \cdots + r_s x_s,$$

with $r_i \in \mathbb{Q}^m$. Now suppose $(A - \mathrm{Id})v = l$, for $A \in \mathrm{GL}(m,\mathbb{Z})$ and $l \in \mathbb{Z}^m$. Then

$$(A-\mathrm{Id})r_0 + x_1(A-\mathrm{Id})r_1 + \cdots + x_r(A-\mathrm{Id})r_s = l,$$

which implies that $x_1(A-\mathrm{Id})r_1 + \cdots + x_r(A-\mathrm{Id})r_s \in \mathbb{Q}^m$. But since $\{1, x_1, \ldots, x_s\}$ is a basis of E over $\mathbb{Q}$, it must follow that

$$x_1(A-\mathrm{Id})r_1 + \cdots + x_r(A-\mathrm{Id})r_s = 0.$$

So if $l \in \mathbb{Z}^m$ and $(A-\mathrm{Id})v = l$, then $(A-\mathrm{Id})r_0 = l$.

We will now check that we cannot have $\mathbb{Z}^m$ contained in the set $Av - v,\ A \in \mathrm{GL}(m,\mathbb{Z})$ if $v \in \mathbb{Q}^m$. In fact, write $(v_1,\ldots,v_m) = (a_1/d,\ldots,a_m/d)$. Then the equation $(A-\mathrm{Id})v = l = (l_1,\ldots,l_m)$ is equivalent to $A(a_1,\cdots,a_m) = (a_1 + l_1 d, \ldots, a_m + l_k d)$. Now choose a prime p not dividing any of the a_i and d. We can solve the equations $a_i + l_i d = 0$ for l_i in $\mathbb{Z}_p$ (since this is a field), which means that each $a_i + l_i d$ will be divisible by p. But the equation $(A-\mathrm{Id})v = l$ then implies that $(a_1,\ldots,a_m) = A^{-1}(a_1 + l_1 d, \ldots, a_m + l_k d)$ is divisible by p as well. So $l \notin \{Av - v,\ A \in \mathrm{GL}(m,\mathbb{Z})\}$. □

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