INTEGRALLY CLOSED FINITE-COLENGTH IDEALS IN TWO-DIMENSIONAL REGULAR LOCAL RINGS ARE MULTIPLIER IDEALS

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Abstract. Multiplier ideals in commutative rings are certain integrally closed ideals with properties that lend themselves to highly interesting applications. But how special are they among integrally closed ideals in general? In this note we show that in a two-dimensional regular local ring with algebraically closed residue field, in the class of ideals containing a power of the maximal ideal there is in fact no difference between "multiplier" and "integrally closed" (or "complete.") However, among "integral" multiplier ideals (or "adjoint ideals") the only simple complete ideals are those of order one.

Introduction. There has arisen in recent years a substantial body of work on "multiplier ideals" in commutative rings (see [La]). Multiplier ideals are integrally closed ideals with properties that lend themselves to highly interesting applications. One is tempted then to ask just how special multiplier ideals are among integrally closed ideals in general.

In this note we show that in a two-dimensional regular local ring R with maximal ideal $\mathfrak{m}$ such that the residue field $R/\mathfrak{m}$ is algebraically closed[1] there is in fact no difference between multiplier ideals and integrally closed ideals, at least when we deal with finite-colength ideals (i.e., those containing a power of $\mathfrak{m}$):

Main Result. *Every integrally closed finite-colength R-ideal is a multiplier ideal.*

We do not know at present whether such a statement holds true for arbitrary integrally closed R-ideals, let alone higher dimensions.

Throughout, $(R, \mathfrak{m})$ will be as above. For convenience we say "complete ideal" instead of "integrally closed finite-colength R-ideal."

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[1]This last condition can most likely be dropped, but we don't want to get involved with the resulting technicalities.

1. Geometric formulation of the problem. The goal of this section is to develop the geometric criterion Corollary 1.4.2 for an ideal to be a multiplier ideal, while laying the groundwork for the proof in the next section that every complete ideal satisfies that criterion.

We begin by recalling some preliminary definitions and known results. (For some historical pointers to the development of the theory of complete ideals see the second paragraph on the first page of [L3].)

For any complete ideal I there exists a *log resolution,* i.e., a proper birational map $f: X \to \mathrm{Spec}(R)$ with X a regular scheme such that the $\mathcal{O}_X$-ideal $I\mathcal{O}_X$ is *invertible.* A quick way to see this, with f a composition of maps obtained by blowing up closed points, is via the Hoskin-Deligne formula [L2, p. 222, Thm. 3.1], which shows that the length of the "transform" of I decreases strictly with each blowup—until it vanishes, at which point I generates an invertible ideal sheaf.

We denote by $E^1, E^2, \dots, E^s$ the integral (i.e., reduced and irreducible) components of the closed fiber $f^{-1}\{\mathfrak{m}\}$. They are isomorphic, as schemes, to the projective line $\mathbb{P}^1_{R/\mathfrak{m}}$, and any two of them intersect transversally. (This can easily be shown by induction on the number of blowups making up f.) Each E^i gives rise to the discrete valuation v_{E^i} whose valuation ring is the local ring on X of the generic point of E^i.

We define the group $\mathrm{Div}_{\mathrm{e}}(X)$ of (exceptional) f*-divisors* to be the free abelian group on the set $\{E^1, \dots, E^s\}$. The invertible sheaf $\mathcal{O}_X(E)$ associated to an f-divisor E is defined in the standard way. We will use repeatedly, without explicit mention, the fact that

$$\Gamma\big(X, \mathcal{O}_X(a_1E^1 + \cdots a_sE^s)\big) = \{\, x \in R \mid v_{E^i}(x) \geq -a_i \text{ for } 1 \leq i \leq s\,\},$$

an R-ideal containing some power of $\mathfrak{m}$.

An f-divisor $E = a_1E^1 + \cdots a_sE^s$ is *effective* if all the integers a_i are ≥ 0, or equivalently, if $\mathcal{O}_X(-E) \subset \mathcal{O}_X$. Such an E can be regarded as a one-dimensional subscheme of X, projective over $R/\mathfrak{m}$, with structure sheaf $\mathcal{O}_E$ fitting in a natural exact sequence

$$0 \to \mathcal{O}_X(-E) \to \mathcal{O}_X \to \mathcal{O}_E \to 0.$$

For any invertible $\mathcal{O}_X$-module $\mathcal{L}$ the *intersection number* $\mathcal{L} \cdot E$ of $\mathcal{L}$ with an effective E is the degree of the invertible $\mathcal{O}_E$-module $\mathcal{L}_E := \mathcal{L} \otimes \mathcal{O}_E$, i.e.,

$$\mathcal{L} \cdot E = \chi_E \mathcal{O}_E - \chi_E \mathcal{L}_E^{-1}$$

where $\chi_E \mathcal{M}$ denotes the Euler characteristic of a coherent $\mathcal{O}_E$-module $\mathcal{M}$ (E being viewed as a projective curve over $R/\mathfrak{m}$.)

For any f-divisor F set $F \cdot E := \mathcal{O}_X(F) \cdot E$. This version of intersection number extends uniquely to a $\mathbb{Z}$-valued symmetric bilinear form on $\mathrm{Div}_{\mathrm{e}}(X)$ (see e.g., [L1, §13]).

An f-divisor F is said to be numerically effective, *nef* for short, if $F \cdot E^i \geq 0$ for all $i = 1, 2, \dots, s$ ($\Rightarrow F \cdot E \geq 0$ for all effective $E \in \mathrm{Div}^+_{\mathrm{e}}(X)$). F is said to be *antinef* if $-F$ is nef.

The following basic result is contained in [L1, p. 220, Thm. (12.1)].

Theorem 1.1. *An f-divisor F is nef if and only if $\mathcal{O}(F)$ is generated by its global sections.*

Corollary 1.1.1. *If F is antinef then F is effective.*

That's because $\mathcal{O}_X(-F) = \Gamma\big(X, \mathcal{O}_X(-F)\big)\mathcal{O}_X \subset \mathcal{O}_X$. (A simpler proof, not using Theorem 1.1, can be found in [L1, p. 238].)

If I is complete and $I\mathcal{O}_X$ is invertible then $I = \Gamma(X, I\mathcal{O}_X)$, whence:

Corollary 1.1.2. (Cf. [L1, §18].) *Sending E to $\Gamma\big(X, \mathcal{O}_X(-E)\big)$ is an isomorphism from the (additive) monoid of antinef f-divisors to the (multiplicative) monoid of those complete R-ideals I such that $I\mathcal{O}_X$ is invertible.*[2]

It is simple to show, by induction on the number of blowups making up f, that the intersection matrix $(E^i \cdot E^j)$ has determinant ± 1. Hence for each i there is a unique f-divisor G_i such that $G_i \cdot E^j = 0$ unless $j = i$, in which case $G_i \cdot E^j = -1$; and for any f-divisor E it holds that $-E = \sum_{i=1}^{s} (E.E^i)G_i$. Thus the monoid of antinef divisors is freely generated by these G_i; in other words, "unique factorization" holds in this monoid—and therefore in the monoid of complete ideals to which, by Corollary 1.1.2, it is isomorphic.

A complete ideal P is *simple* if whenever $P = IJ$ then either I or J is the unit ideal.

Corollary 1.1.3. (Zariski, [ZS, p. 386, Thm. 3].) *Every complete ideal is, in a unique way, the product of simple complete ideals.*

(For deducing this corollary it helps to note that for any product IJ of ideals, if $IJ\mathcal{O}_X$ is invertible then $I\mathcal{O}_X$ and $J\mathcal{O}_X$ are both invertible.)

Corollary 1.1.4. *If I is a complete ideal such that $I\mathcal{O}_X = \mathcal{O}_X(-E)$ is invertible and if $P_i := \Gamma\big(X, \mathcal{O}_X(-G_i)\big)$ is the simple complete ideal corresponding to the above G_i, then $I = \prod_{i=1}^{s} P_i^{E.E^i}$. In particular, P_i divides $I \iff E \cdot E^i \neq 0$.*

Moreover, the valuations v_{E^i} associated to those E^i such that $E \cdot E^i \neq 0$ are precisely the Rees valuations of I (i.e., those valuations whose valuation ring is the local ring of the generic point of some integral component of the closed fiber of the normalized blowup of I). (See [L4, p. 300, Prop. (4.4)].)

The following Lemma[3] will be needed. We write $\sum_i a_i E^i \geq \sum_i b_i E^i$ if $a_i \geq b_i$ for all $i = 1, 2, \ldots, s$.

Lemma 1.2. *Let $E = \sum_i a_i E^i$ and let I be the R-ideal $\Gamma(X, \mathcal{O}_X(-E))$. Then $I\mathcal{O}_X$ is invertible. Equivalently, there exists an antinef $E^- \geq E$ such that $I = \Gamma(X, \mathcal{O}_X(-E^-))$.*

Corollary 1.1.2 implies that this E^- must be the least antinef f-divisor $\geq E$.

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[2]It is a theorem of Zariski that a product of two complete ideals is still complete [ZS, p. 385, Thm. 2′], [L1, p. 209, Thm. (7.1)].

[3]related to Enriques's "principle of discharge" [Z, p. 28],

Proof. One may assume $X \neq \mathrm{Spec}(R)$. There exists *some* antinef $F \geq E$: pick n such that $\mathfrak{m}^n \subset I$, then define F by $\mathfrak{m}^n\mathcal{O}_X = \mathcal{O}_X(-F)$. Among all antinef $F = \sum b_iE^i \geq E$ choose one—call it F_E—for which $\sum_i(b_i - a_i)$ has minimal value, denoted σ_E.

Procede by induction on σ_E. Suppose $\sigma_E > 0$ (otherwise there is nothing to prove), and that the Lemma holds for all E' with $\sigma_{E'} < \sigma_E$. With $F := F_E$ as above, there is an i such that $E \cdot E^i > 0 \geq F \cdot E^i$, and since $E^j \cdot E^i \geq 0$ when $j \neq i$ (clearly) and $E^i \cdot E^i < 0$,[4] therefore $b_i > a_i$, whence $F \geq E + E^i$. So $\sigma_{E+E^i} < \sigma_E$, and therefore $\Gamma(X, \mathcal{O}_X(-E-E^i))\mathcal{O}_X$ is invertible. It suffices then to verify that $\Gamma\big(X, \mathcal{O}_X(-E-E^i)\big) = I$, by applying the left-exact functor $\Gamma(X, -)$ to the natural exact sequence

$$0 \longrightarrow \mathcal{O}_X(-E-E^i) \longrightarrow \mathcal{O}_X(-E) \longrightarrow \mathcal{O}_X(-E) \otimes \mathcal{O}_{E^i}$$

and observing that since $\mathcal{O}_X(-E) \otimes \mathcal{O}_{E^i}$ has degree $-E \cdot E^i < 0$, therefore

$$\Gamma\big(X, \mathcal{O}_X(-E) \otimes \mathcal{O}_{E^i}\big) = \Gamma\big(E^i, \mathcal{O}_X(-E) \otimes \mathcal{O}_{E^i}\big) = 0. \qquad \square$$

(1.3). (Canonical divisors.) Let $Y \xrightarrow{g} X \xrightarrow{f} \mathrm{Spec}(R)$ be proper birational maps with Y and X regular schemes. By a theorem of Zariski and Abhyankar (see, e.g., [L1, p. 204, Thm. (4.1)]) both f and g are compositions of point blowups.

Let $F^1, F^2, \dots, F^t$ be the integral components of $(fg)^{-1}\{\mathfrak{m}\}$. For an f-divisor E, g^*E denotes the fg-divisor with F^i-coefficient $v_{F^i}(\mathcal{O}_X(-E)_{x_i})$, where $x_i \in X$ is the g-image of the generic point of F^i.

Since, as before, the intersection matrix $(F^i \cdot F^j)$ has determinant ± 1, there is a unique fg-divisor K_g such that

$$K_g \cdot F^i = \begin{cases} -F^i \cdot F^i - 2 & \text{if } g(F^i) \text{ is a point} \\ 0 & \text{otherwise.} \end{cases}$$

This K_g is called the *canonical divisor of* g.[5]

(1.3.1). The following easily-checked properties characterize K_g for all g:

- If g is the blowup of a closed point $x \in X$ then $K_g = g^{-1}\{x\}$.
- If $Z \xrightarrow{h} Y \xrightarrow{g} X \xrightarrow{f} \mathrm{Spec}(R)$ are proper birational maps with Z, Y and X regular schemes, then

$$K_{gh} = h^*K_g + K_h.$$

(1.4). (Multiplier ideals.) For $E = \sum_i a_iE^i \in \mathrm{Div_e}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ set

$$[E] = \sum_i [a_i]E^i \in \mathrm{Div_e}(X)$$

where $[a_i]$ is the greatest integer $\leq a_i$.

[4] It is well-known, going back to Du Val, that the intersection matrix $(E^i \cdot E^j)$ is negative definite, see e.g., [L1, p. 224, Lemma (14.1)].

[5] One has $\mathcal{O}(K_g) = H^0(g^!\mathcal{O}_X)$ with $g^!$ as in Grothendieck duality theory [LS, p. 206, (2.3)].

Definition 1.4.1. Let I be a complete R-ideal, $h : Y \to \operatorname{Spec}(R)$ a log resolution of I, say $I\mathcal{O}_Y = \mathcal{O}_Y(-G)$, and let c be a positive real number. The *multiplier ideal* $\mathcal{J}(cI)$ is defined to be

$$\mathcal{J}(cI) := \Gamma\big(Y, \mathcal{O}(K_h - [cG])\big).$$

Thus, by Lemma 1.2,

$$\mathcal{J}(cI)\mathcal{O}_Y = \mathcal{O}_Y\big(-D\big)$$

where $D := \big([cG] - K_h\big)^{-}$ is the least antinef h-divisor $\geq [cG] - K_h$.

For a point blowup $h_1 : Y_1 \to Y$ one finds via (1.3.1) that the log resolution h can be replaced by the log resolution $h \circ h_1$ without affecting $\mathcal{J}(cI)$. Since any two log resolutions are dominated by a third, obtained from each of the two by a sequence of blowups,[6] it follows that $\mathcal{J}(cI)$ does not depend on the choice of the log resolution h.

Corollary 1.4.2. *A complete R-ideal J satisfies $J = \mathcal{J}(cI)$ for some c, I iff for some log resolution $h : Y \to \operatorname{Spec}(R)$ of J, say $J\mathcal{O}_Y = \mathcal{O}_Y(-F)$, there is an antinef h-divisor G and a real $c > 0$ such that*

$$F = \big([cG] - K_h\big)^{-}. \tag{1.4.2.1}$$

2. Proof of Main Result. Let J be a complete R-ideal. We will describe a log resolution $h : Y \to \operatorname{Spec}(R)$ of J, and a G and c as in Corollary 1.4.2, such that if $J\mathcal{O}_Y = \mathcal{O}_Y(-F)$ then (1.4.2.1) holds. (The number of suitable (h, G, c) will be enormous.)

Factor J as $J = \prod_{\ell=1}^{u} P_\ell^{e_\ell}$ (P_ℓ simple complete, $e_\ell > 0$)—see Corollary 1.1.3 and the two paragraphs preceding it. Let $f : X \to \operatorname{Spec}(R)$ be any log resolution of J, $J\mathcal{O}_X = \mathcal{O}_X(-F^0)$. We construct a proper birational map $g_N : Y_N \to X$, with Y_N regular, for each u-tuple $N := (n_1, n_2, \dots, n_u)$ of non-negative integers, as follows.

For convenience of expression, we say "blow up a closed point $x \in X$ generically, n times" to mean "blow up $x_0 := x$ to get $g_1 : X_1 \to X$, then blow up a closed point x_1 on $g_1^{-1}x_0$ but not on any other integral component of the closed fiber of $X_1 \to \operatorname{Spec}(R)$ to get $g_2 : X_2 \to X_1$, then blow up a closed point x_2 on $g_2^{-1}x_1$ but not on ... then blow up a closed point x_{n-1} on $g_{n-1}^{-1}x_{n-2}$ but not on any other integral component of the closed fiber of $X_{n-1} \to \operatorname{Spec}(R)$ to get $g_n : X_n \to X_{n-1}$."

Then with $g := g_1 \circ g_2 \circ \circ \cdots \circ g_n$ it holds that:

(2.1) $g^{-1}x$ is a chain of n integral curves $D_1, D_2, \dots, D_n$ such that for $0 < i < n$, $D_i \cdot D_{i+1} = 1$ and $D_i \cdot D_i = -2$, while $D_n \cdot D_n = -1$; and if $|j - i| > 1$ then $D_i \cdot D_j = 0$.

[6] By the above-mentioned theorem of Zariski and Abhyankar, it suffices to principalize some ideal sheaf on one of the log resolutions by a sequence of point blowups ("elimination of indeterminacies"), which can be done e.g., via the Hoskin-Deligne formula, as before.

(For the proof one can use, e.g., [L1, p. 229, middle, and p. 227, α) and β)]. Here, and subsequently, the reader may find it useful to do some rough sketches.)

As before, there corresponds to each D_i a simple complete ideal P_i; and, we claim, *these P_i form a strictly decreasing sequence $P_1 > P_2 > \cdots > P_n$*, with P_1 strictly contained in each of the simple ideals corresponding to the (one or two) integral components E^j of $f^{-1}\{\mathfrak{m}\}$ passing through x.

Indeed, let G_j be the f-divisor such that $G_j \cdot E^j = -1$ and $G_j \cdot E = 0$ for every other integral component E of $f^{-1}\{\mathfrak{m}\}$, and let Q_j be the corresponding simple complete ideal. It follows from, e.g., [L1, p. 227, α) and β)] that $g_1^*G_j$ is antinef; and the corresponding simple complete ideal is

$$\Gamma\big(X_1, \mathcal{O}_{X_1}(-g_1^*G_j)\big) = \Gamma\big(X, \mathcal{O}_X(-G_j)\big) = Q_j.$$

Further, with $E' := g_1^{-1}x$, let G' be the fg_1-divisor such that $G' \cdot E' = -1$ and $G' \cdot E'' = 0$ for every other integral component E'' of $(fg_1)^{-1}\{\mathfrak{m}\}$. Then $g_1^*G_j + E'$ has intersection number -1 with E' and ≥ 0 with each E''. So $G' - g_1^*G_j - E'$ is antinef, hence effective (Corollary 1.1.1); and consequently $G' > g_1^*G_j$. Thus the simple complete ideal $\Gamma\big(X_1, \mathcal{O}_{X_1}(-G')\big)$ is strictly contained in Q_j.

Continuing in this way we establish the claim.

Now for each simple factor P_ℓ of J, there is a unique integral component E^ℓ of $f^{-1}\{\mathfrak{m}\}$ such that $P_\ell\mathcal{O}_X \cdot E^\ell = 1$ and $P_\ell\mathcal{O}_X \cdot E = 0$ for any other integral component E; and hence $e_\ell = J\mathcal{O}_X \cdot E^\ell$. For each $\ell = 1, 2, \ldots, u$, pick e_ℓ distinct closed points $x_1, \ldots, x_{e_\ell}$ which lie on E^ℓ but on no $E \neq E^\ell$ and blow up all of these points generically, n_ℓ times. Then Y_N is the resulting surface, and g_N is the composition of all the blowups. It is easily seen that (Y_N, g_N) does not depend (up to isomorphism) on the order in which the chosen points are blown up—though that won't really be used.[7]

To simplify notation, fix N and set $(Y, g) := (Y_N, g_N)$ and $F := g^*F^0$, so that $J\mathcal{O}_Y = \mathcal{O}_Y(-F)$. Also, set $h := fg : Y \to \operatorname{Spec}(R)$.

For an f-divisor E, we denote by E^* the proper transform of E on Y, i.e., if $g^*E = \sum_i a_iE_i' + \sum_j b_jE_j''$ where each gE_i' is a curve on X while each gE_j'' is a closed point, then $E^* := \sum_i a_iE_i'$.

For each $\ell = 1, 2, \ldots, u$ and $x_{j_\ell} \in E^\ell$ $(j_\ell = 1, 2, \ldots, e_\ell)$ let

$$\{\, E^\ell_{j_\ell k_\ell} \mid k_\ell = 1, 2, \ldots, n_\ell \,\}$$

be the family of integral curves on Y whose g-image is x_{j_ℓ}, the ordering of these curves by the index k_ℓ conforming to the ordering of the D's in (2.1).

[7]The initial $\sum_\ell e_\ell$ points could be taken to be the intersection of the closed fiber on X and a generic curve C in the linear system $|-F|$ (i.e., a divisor—having no component in the closed fiber—of the form $(j) - F$ with j a generic element of J). Then at each stage the points to be blown up could be taken to be closed points on the inverse image of C.

If a_ℓ is the E^ℓ-coefficient of the divisor F^0, and b_ℓ of the divisor K_f, then one finds (using (1.3.1)) that

$$\begin{aligned} F = g^*F^0 &= F^{0*} + \sum_\ell \sum_{j_\ell, k_\ell} a_\ell E^\ell_{j_\ell k_\ell}, \\ g^*K_f &= K_f^* + \sum_\ell \sum_{j_\ell, k_\ell} b_\ell E^\ell_{j_\ell k_\ell}, \\ K_g &= \sum_\ell \sum_{j_\ell, k_\ell} k_\ell E^\ell_{j_\ell k_\ell}. \end{aligned}$$

Set $G := F + K_g$. Expanding F and K_g as above, noting that $g^*D \cdot E^\ell_{j_\ell k_\ell} = 0$ for any f-divisor D [L1, p. 227, β)], and using (2.1), one calculates that for every integral component E of $h^{-1}\{\mathfrak{m}\}$, $G \cdot E = 0$ unless E is one of the curves $E^\ell_{j_\ell n_\ell}$ at the end of the chains emanating from the $\sum_\ell e_\ell$ originally chosen points (i.e., $g(E)$ is a point and $E \cdot E = -1$), in which case $G \cdot E = -1$. So G is antinef, and by Corollary 1.1.4,

$$I := \Gamma\big(Y, \mathcal{O}_Y(-G)\big)$$

is the product of the simple complete ideals corresponding to these $\sum_\ell e_\ell$ curves having self-intersection -1.

Here is the key technical point:

Lemma 2.2. *For all sufficiently small $\epsilon > 0$, there exists N such that*

$$(1+\epsilon)G - K_h = F + A \qquad \big([A] \le 0\big) \tag{2.2.1}$$

where the coefficient of $[A]$ at each $E^\ell_{j_\ell n_\ell}$ is 0.

Proof. Using (1.3.1), one transforms (2.2.1) into the equality

$$\epsilon(F + K_g) - g^*K_f = A.$$

More explicitly (see above)

$$\epsilon\Big(F^{0*} + \sum_\ell \sum_{j_\ell, k_\ell} (a_\ell + k_\ell) E^\ell_{j_\ell k_\ell}\Big) - \Big(K_f^* + \sum_\ell \sum_{j_\ell, k_\ell} b_\ell E^\ell_{j_\ell k_\ell}\Big) = A.$$

So to get (2.2.1) we can choose any $\epsilon > 0$ such that the coefficients of $\epsilon F^0 - K_f$ are all < 1, and then look for n_ℓ such that $\epsilon(a_\ell + k_\ell) - b_\ell < 1$ for all ℓ and $k_\ell \le n_\ell$, while $\epsilon(a_\ell + n_\ell) - b_\ell \ge 0$. These conditions mean precisely that n_ℓ satisfies the inequalities

$$1/\epsilon + b_\ell/\epsilon - a_\ell > n_\ell \ge b_\ell/\epsilon - a_\ell \quad (\ell = 1, 2, \dots, u).$$

Clearly, such integers n_ℓ can be found if $\epsilon < 1$. $\square$

For $c = 1+\epsilon$ and N satisfying Lemma 2.2, and with $h\colon Y \to \mathrm{Spec}(R)$ and F, G, as before, we have

$$F' := \big([cG] - K_h\big)^{-} \le F,$$

so that

$$J' := \Gamma\big(Y, \mathcal{O}(-F')\big) \supset J = \Gamma\big(Y, \mathcal{O}(-F)\big).$$

Let us verify that $J' = J$, thereby proving the main result.

Recall that the valuations $v_\ell := v_{E^\ell}$ are just the Rees valuations of J. (See the remark following Corollary 1.1.4). So

$$J = \{\, \rho \in R \mid v_\ell(\rho) \ge v_\ell(J) \text{ for all } \ell = 1, 2, \dots, u \,\}.$$

Thus we need only show that for each ℓ, the $E^{\ell *}$-coefficient a'_ℓ of F' is the same as that of F (namely a_ℓ). Let us say that ℓ is "good" if $a'_\ell = a_\ell$ and "bad" if $a'_\ell < a_\ell$.

If ℓ is good then since $F' \le F$ therefore $F' \cdot E^{\ell *} \le F \cdot E^{\ell *}$. Corollary 1.1.4 shows then that J' is divisible by $P_\ell^{e_\ell}$.

Suppose ℓ is bad. For $j \in [1, e_\ell]$ and with a'_{jk} the E^ℓ_{jk}-coefficient of F' it is easily seen that

$$a'_\ell =: a'_{j0} \le a'_{j1} \le a'_{j2} \le \cdots \le a'_{jn_\ell} = a_\ell.$$

So there is a $k \in [1, n_\ell]$ such that $a'_{j,k-1} < a'_{jk} = a'_{j,k+1} = \cdots = a_\ell$. Then

$$F' \cdot E^\ell_{kl} = \begin{cases} a'_{j,k-1} - 2a'_{jk} + a'_{j,k+1} < 0 & \text{if } k < n_\ell, \\ a'_{j,n_\ell - 1} - a'_{jn_\ell} < 0 & \text{if } k = n_\ell. \end{cases}$$

From Corollary 1.1.4 and the remarks after 2.1, one deduces that J' is divisible by a simple complete ideal $P'_{\ell j} < P_\ell$. This being so for all j, and the $P'_{\ell j}$ being distinct (Corollary 1.1.2), it follows from Corollary 1.1.3 that J' is divisible by $P'_{\ell 1} P'_{\ell 2} \cdots P'_{\ell e_\ell} < P_\ell^{e_\ell}$.

Thus (by Corollary 1.1.3 again) the existence of a bad ℓ leads to a factorization of J' which contradicts $J' \supset J$. So every ℓ is good, and $J' = J$. $\square$

Remarks. 1. By the choice of ϵ, the E^ℓ-coefficient of $\epsilon F^0 - K_f$ is < 1, i.e., $\epsilon a_\ell - b_\ell < 1$, i.e., $b_\ell/\epsilon - a_\ell > -1/\epsilon$. It could happen that $b_\ell/\epsilon - a_\ell < 0$ for all ℓ. In this case one can take $N = (0, 0, \dots, 0)$, and then $J = \mathcal{J}((1+\epsilon)J)$.

2. The proof shows that if J is a simple complete ideal then there is a simple complete ideal $P \subset J$ and a $c > 0$ such that $J = \mathcal{J}(cP)$.

3. Since the $c = 1+\epsilon$ we have considered can be arbitrarily close to 1, one might ask if it is possible for c actually to be 1. (This would be the case studied in [L5], where $\mathcal{J}(I)$ is called the *adjoint ideal* of I.)

For simple complete J, the answer is given by:

Proposition 2.3. *A simple complete ideal J is of the form $\mathcal{J}(I)$ for some I $\iff$ $J \not\subset \mathfrak{m}^2$ $\iff$ $J = (a, b^n)R$ for some integer $n > 0$ and $a, b \in R$ such that $(a, b)R = \mathfrak{m}$.*

Proof. The last $\iff$ holds because $J \not\subset \mathfrak{m}^2$ means that J contains an element a such that R/aR is a discrete valuation ring. Moreover, if $(a, b)R = \mathfrak{m}$ and $z \in R$ is integral over $J = (a, b^n)R$ then the canonical image of z in R/aR is integral over—and hence is a multiple of—that of b^n, whence $z \in J$, and thus J is complete (and clearly simple). And it is a simple exercise to show that for such a J, $J = \mathcal{J}(J^2)$. (One could use [L5, p. 749, Prop. (3.1.2)].)

For any simple complete J, on the minimal log resolution $f : X \to \operatorname{Spec}(R)$ there is, as we have seen, a unique integral component E of $f^{-1}\{\mathfrak{m}\}$ such that $J\mathcal{O}_X \cdot E \neq 0$. This E must satisfy $E \cdot E = -1$: for, there is at least one integral component E' such that $E' \cdot E' = -1$, namely the closed fibre $f'^{-1}\{x'\}$ for the blowup $f' : X \to X'$ of $x' \in X'$ coming last in some seqence of blowups composing to f; and by minimality of f, $J' := J\mathcal{O}_{X',x'}$ cannot be invertible, from which one sees, with $\mathfrak{m}'$ the maximal ideal of $R' := \mathcal{O}_{X',x'}$, that $J' = d\mathfrak{m}'^t$ for some $d \in R'$ and $t > 0$, whence $J\mathcal{O}_X \cdot E' = t$. (Of course $t = 1$.)

We claim that if $F \cdot F = -2$ for all integral components $F \neq E$ of $f^{-1}\{\mathfrak{m}\}$, then $J \not\subset \mathfrak{m}^2$.[8] Indeed, this condition on the F's means that among the base points of J no two are proximate to the same one, and the conclusion follows from [L5, p. 301, (3)].

Assuming now that our simple $J = \mathcal{J}(I)$, let $h : Y \to \operatorname{Spec}(R)$ be the minimal log resolution of I, say $I\mathcal{O}_Y = \mathcal{O}_Y(-G)$. As above, we find that for any integral component F of $h^{-1}\{\mathfrak{m}\}$, $F \cdot F = -1 \Rightarrow G \cdot F < 0$. (We may assume that F is the closed fiber for the last blowup in some sequence of blowups composing to g.) Since $K_h \cdot F = -F \cdot F - 2 = -1$ for every such F, it follows that $G - K_h$ is antinef, and hence $\mathcal{O}_Y(K_h - G) = J\mathcal{O}_Y$.

Since $J\mathcal{O}_Y$ is invertible (Lemma 1.2), the minimality of f implies that there exists a $g : Y \to X$ such that $h = fg$. Let $F^{1*}, F^{2*}, \ldots, F^{n*}$ be the proper transforms on Y of the integral components $F^1, F^2, \ldots, F^n$ of $f^{-1}\{\mathfrak{m}\}$ other than E (see above). Then $F^{i*} \cdot F^{i*} \leq F^i \cdot F^i \leq -2$ and so $K_h \cdot F^{i*} \geq 0$. But (by [L1, p. 227, β)])

$$K_h \cdot F^{i*} \leq (K_h - G) \cdot F^{i*} = JO_Y \cdot F^{i*} = JO_X \cdot F^i = 0,$$

and thus $K_h \cdot F^{i*} = 0$, i.e., $F^{i*} \cdot F^{i*} = -2$, whence, finally, $F^i \cdot F^i = -2$. The above claim shows then that $J \not\subset \mathfrak{m}^2$. $\square$

[8] The converse is part of the exercise at the end of the first paragraph.

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