ESTIMATES FOR THE $\bar{\partial}$-NEUMANN PROBLEM AND NONEXISTENCE OF LEVI-FLAT HYPERSURFACES IN $\mathbb{C}P^n$

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Abstract. Let Ω be a *pseudoconvex* domain with C^2-smooth boundary in $\mathbb{C}P^n$. We prove that the $\bar{\partial}$-Neumann operator N exists for (p,q)-forms on Ω. Furthermore, there exists a $t_0 > 0$ such that the operators N, $\bar{\partial}^* N$, $\bar{\partial} N$ and the Bergman projection are regular in the Sobolev space $W^t(\bar{\Omega})$ for $t < t_0$.

The boundary estimates above have applications in complex geometry. We use the estimates to prove the nonexistence of $C^{2,\alpha}$ real Levi-flat hypersurfaces in $\mathbb{C}P^n$. We also show that there exist no non-zero L^2-holomorphic $(p,0)$-forms on any *pseudoconcave* domain in $\mathbb{C}P^n$ with $p > 0$.

Introduction

One of the main results in this paper is the following:

Theorem 1. *There exists no $C^{2,\alpha}$ real Levi-flat hypersurface M^{2n-1} in $\mathbb{C}P^n$, where $0 < \alpha < 1$ and $n \geq 2$.*

Theorem 1 is inspired by the recent papers of Siu [Siu2,3] who proved that there exists no C^8 Levi-flat hypersurface in $\mathbb{C}P^n$, $n \geq 2$. The required smoothness has been reduced to C^4 by Iordan [Io]. Nonexistence of real analytic Levi-flat hypersurfaces in $\mathbb{C}P^n$ was obtained in Lins Neto [LNe] for $n \geq 3$ and Ohsawa [Oh] for $n = 2$. Our proof of Theorem 1 follows arguments along the lines of [Siu2,3] who reduced the proof of Theorem 1 to the regularity of the tangential Cauchy-Riemann equations on M.

We derive new boundary regularity results for $\bar{\partial}$-equation and the $\bar{\partial}$-Neumann problem for domains in $\mathbb{C}P^n$. Our Theorem 2 stated below can be applied to *not only* domains Ω with Levi-flat boundaries M *but also all* other weakly pseudoconvex C^2 domains in $\mathbb{C}P^n$, when compared with the earlier work mentioned above. The boundary regularity of the solutions is interesting itself.

To prove Theorem 1, Siu [Siu2,3] made the following observation: If there exists a C^2-smooth *real* Levi-flat hypersurface $M = M^{2n-1}$ in $\mathbb{C}P^n$ and if $\mathbb{C}P^n$ has the

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standard Fubini-Study metric, then the curvature form $i\tilde{\Theta}^N$ of its complex normal line bundle $\mathcal{N}_{M,\mathbb{C}P^n}$ is *strictly positive definite* on $T^{(1,0)}(M) \oplus T^{(0,1)}(M)$ by the Cartan-Chern-Gauss structure equation.

Furthermore, one has $\tilde{\Theta}^N = d\theta$ is an exact form, where θ is the connection form of the complex normal line bundle $\mathcal{N}_{M,\mathbb{C}P^n}$. If M is a Levi-flat hypersurface, it is locally foliated by complex manifolds of complex dimension $n-1$. It is known that the restriction of $\tilde{\Theta}^N$ to each complex leaf of M, $\Theta_b = \tilde{\Theta}^N|_{[T(M)]_{\mathbb{R}} \cap J[T(M)]_{\mathbb{R}}}$ is a $(1,1)$-form, where J is the complex structure of $\mathbb{C}P^n$ (see Proposition 1.2 below).

On the other hand, if one could show that Θ_b is $\partial_b\bar{\partial}_b$-exact for some continuous real-valued function h (i.e., $\Theta_b = \partial_b\bar{\partial}_b h$ on M), then $i\Theta_b$ is *non-positive* at the maximum point of h on M (cf. [Ca]). This contradicts the fact that $i\tilde{\Theta}^N$ is *strictly positive definite* on $T^{(1,0)}(M) \oplus T^{(0,1)}(M)$, and hence Theorem 1 would follow immediately.

Therefore, the proof of Theorem 1 is reduced to a problem of finding a continuous function u satisfying

$$i\partial_b\bar{\partial}_b u = f_b = f|_{[T(M)]_{\mathbb{R}} \cap J[T(M)]_{\mathbb{R}}} \quad \text{on } M, \tag{0.1}$$

under the condition that $f = d\theta$ is an exact real-valued (1, 1)-form when restricted to $T^{(1,0)}(M) \oplus T^{(0,1)}(M)$. Equation (0.1) corresponds to the classical $\partial\bar{\partial}$-Lelong equation

$$i\partial\bar{\partial}\tilde{u} = \tilde{f} \quad \text{in } \mathbb{C}P^n \tag{0.2}$$

where $\tilde{f} = d\tilde{\theta}$ is an exact real-valued (1, 1)-form. Using the fact that $\tilde{f}$ is an exact real-valued (1,1)-form to solve equation (0.2), it suffices to solve

$$\bar{\partial}\tilde{u} = \tilde{\theta}^{(0,1)} \quad \text{in } \mathbb{C}P^n, \tag{0.3}$$

where $\tilde{\theta}^{(0,1)}$ is the $(0,1)$ part in $\tilde{\theta}$, which is $\bar{\partial}$-closed in $\mathbb{C}P^n$. By the Hodge theory, there is no nontrivial harmonic $(0,1)$-form in $\mathbb{C}P^n$. It follows that any $\bar{\partial}$-closed (0,1)-form $\tilde{\theta}^{(0,1)}$ on $\mathbb{C}P^n$ must be $\bar{\partial}$-exact, and hence equation (0.3) can be solved easily (e.g., cf. [Zh]). For the proof of Theorem 1 and equation (0.1), one can similarly deduce that it suffices to solve

$$\bar{\partial}_b u = \theta_b \quad \text{in } M, \tag{0.4}$$

where θ_b is a $(0,1)$-form in M satisfying some compatibility condition. In [Siu2,3], existence and regularity of equation (0.4) on a Levi-flat boundary M in $\mathbb{C}P^n$ is studied. In this paper, we study the more general situation when M is the C^2 boundary of any pseudoconvex domain Ω in $\mathbb{C}P^n$.

When $n > 2$, the compatibility condition for equation (0.4) to be solvable is that

$$\bar{\partial}_b\theta_b = 0 \quad \text{on } M. \tag{0.5}$$

When $n = 2$, (0.5) is satisfied trivially and the compatibility condition for equation (0.4) is substituted by the following moment condition:

$$\int_M \theta_b \wedge \Psi = 0, \tag{0.6}$$

where Ψ is any $\bar\partial_b$-closed (2,0)-form on M (see (9.2.12a) of [CS, p216].) To show that such a compatibility condition holds for our case with $n = 2$, we also derived several Liouville type theorems which are of independent interest. In particular, we show that on any *pseudoconcave* domain Ω with C^2 boundary in $\mathbb{C}P^n$, there exist no non-zero L^2-integrable holomorphic $(p, 0)$-forms on Ω with $p > 0$, (cf. Proposition 4.5 below).

If one can extend a $(0, 1)$-form θ_b satisfying (0.5) or (0.6) from the boundary $M = b\Omega$ to a $\bar\partial$-closed form $\tilde\theta$ on the domain Ω, then (0.4) can be solved by restricting the solution $\tilde u$ of the $\bar\partial$-equation (0.3) to $M = b\Omega$. To study the $\bar\partial$-closed extension from M to Ω, one can formulate it as a $\bar\partial$-Cauchy problem of finding solutions to the $\bar\partial$-equation (0.3) with prescribed compact support. When Ω is strictly pseudoconvex with smooth boundary, the $\bar\partial$-closed extension on Ω was pioneered in the work of Kohn and Rossi [KoR] using the boundary regularity of the $\bar\partial$-Neumann operator on strongly pseudoconvex domains. When $\Omega \subset \mathbb{C}^n$ is only pseudoconvex, the $\bar\partial$-closed extension for forms on $M = b\Omega$ to Ω can also be obtained if one can obtain the boundary regularity for the $\bar\partial$-Neumann problem or weighted $\bar\partial$-Neumann problem (see [Sh,BSh] or [CS, Chap.9]).

In our case, the connected real hypersurface M divides $\mathbb{C}P^n$ into two connected domains: Ω_+ and Ω_-. Both Ω_+ and Ω_- are pseudoconvex and have Levi-flat boundaries. Using the boundary regularity of the $\bar\partial$-Neumann operator, a two-sided $\bar\partial$-closed extension $\tilde\theta_\pm$ will be constructed for any form θ_b satisfying the compatibility condition (0.5) or (0.6) on the Levi-flat hypersurface M, where the $\bar\partial$-closed extension $\tilde\theta = \tilde\theta_\pm$ is defined on the whole space $\mathbb{C}P^n = \overline{\Omega_+ \cup \Omega_-}$.

We let $N = N_{(p,q)}$ denote the $\bar\partial$-Neumann operator on Ω, i.e., the inverse operator for the Laplace operator $\Box_{(p,q)}$ acting on (p, q)-forms with *the $\bar\partial$-Neumann boundary condition* on Ω. Let $W^s_{(p,q)}(\Omega)$ be the space of all (p, q)-forms whose coefficients are in the Sobolev space $W^s(\Omega)$. To prove Theorem 1, we first show that $N = N_{(p,q)}|_\Omega$ exists and that the operators N, $\bar\partial^* N$ and $\bar\partial N$ are bounded on W^s for some $s > 0$, when $\Omega \subset \mathbb{C}P^n$ is a pseudoconvex domain with C^2 boundary.

To formulate our main regularity result for the $\bar\partial$-Neumann operator N, we introduce a geometric invariant for any pseudoconvex domain $\Omega \subset \mathbb{C}P^n$ as follows:

Definition 0.1. *Let Ω be a pseudoconvex domain in a Kähler manifold with C^2-smooth boundary. Let $\delta(x) = \delta_{b\Omega}(x) = d(x, b\Omega)$ be a distance function from the boundary. We call $t_0 = t_0(\Omega)$ the order of plurisubharmonicity for the distance function $\delta_{b\Omega}$ if*

$$t_0(\Omega) = \sup\{0 < \epsilon \le 1 | i\partial\bar\partial(-\delta^\epsilon) \ge 0 \ on\ \Omega\}. \tag{0.7}$$

When $t_0(\Omega) = 1$, the condition implies that there exists a plurisubharmonic defining function on Ω. For a general smooth pseudoconvex domain in $\mathbb{C}^n$, such a plurisubharmonic defining function does not necessarily exist (cf. [DF2]). However, Diederich-Fornaess [DF1] showed that there exists a $0 < t_0(\Omega) \leq 1$ for any pseudoconvex domain Ω in $\mathbb{C}^n$ with C^2-smooth boundary (for a simple proof for pseudoconvex domains in $\mathbb{C}^n$ with C^3 boundary, see Range [R]). In $\mathbb{C}P^n$, Ohsawa-Sibony [OS] showed that there exists $0 < t_0(\Omega) \leq 1$ for any pseudoconvex domain $\Omega \subset \mathbb{C}P^n$ with C^2-smooth boundary using results of Takeuchi [Ta] and [DF1].

Theorem 2. *Let Ω be a pseudoconvex domain with C^2-smooth boundary in $\mathbb{C}P^n$ and let $t_0(\Omega)$ be the order of plurisubharmonicity for the distance function $\delta_{b\Omega}$. Then the $\bar\partial$-Neumann operator $N_{(p,q)}$ exists on $L^2_{(p,q)}(\Omega)$ where $0 \leq p, q \leq n$. Furthermore, $N, \bar\partial N, \bar\partial^* N$ and the Bergman projection P are exact regular operators on $W^s_{(p,q)}(\Omega)$ for $0 < s < \frac{1}{2}t_0$ with respect to the $W^s(\Omega)$-Sobolev norms.*

The Bergman projection P is the orthogonal projection from $L^2_{(p,q)}(\Omega)$ to $ker(\bar\partial)$. In general, P or the $\bar\partial$-Neumann operator N is not necessarily a bounded operator from $W^s_{(p,q)}(\Omega)$ to $W^s_{(p,q)}(\Omega)$ for smooth pseudoconvex domains Ω (cf. [Ba]). In fact, Barrett [Ba] showed that, for any given β with $0 < \beta < 1$, there is a pseudoconvex domain Ω_β (the so-called Diederich-Fornaess' worm domain [DF2]) with C^∞-smooth boundary such that the Bergman projection P and the $\bar\partial$-Neumann operator N on Ω_β are *not bounded* from $W^\beta_{(0,1)}(\Omega_\beta)$ to itself. Christ [Chr] showed that P does *not* map $C^\infty(\bar\Omega_\beta)$ into $C^\infty(\bar\Omega_\beta)$ for such a domain Ω_β. Therefore, one can only expect the boundary regularity of the $\bar\partial$-Neumann operator N to be regular in $W^s_{(p,q)}(\Omega)$ with small s for general pseudoconvex domains.

When $\Omega \subset\subset \mathbb{C}^n$ with C^∞-smooth boundary and Ω has a plurisubharmonic defining function, Boas and Straube [BS] prove that the conclusion of Theorem 2 holds for any $s > 0$. This corresponds to the case when $t_0(\Omega) = 1$. For the case $t_0(\Omega) < 1$ and $\Omega \subset \mathbb{C}^n$, Kohn [Ko2] and Berndtsson-Charpentier [BC] obtain the Sobolev regularity for the operators $\bar\partial^* N$ and the Bergman projection P. Boas-Straube [BS1] show that regularity for the Bergman projection and the $\bar\partial$-Neumann operator are equivalent for C^∞-smooth pseudoconvex domains in $\mathbb{C}^n$ (and also for smooth pseudoconvex domains in complex manifolds with strongly plurisubharmonic functions in a neighborhood of the boundary). For pseudoconvex domains $\Omega \subset \mathbb{C}P^n$, such a smooth strictly plurisubharmonic function ϕ might not exist in a neighborhood of the boundary $b\Omega$. Therefore, the proof of Theorem 2 has extra difficulties when compared to the case of domains in $\mathbb{C}^n$. We remark that our proof of the regularity for the Bergman projection and $\bar\partial^* N$ is similar to the proof in [BC], but the regularity results of the $\bar\partial$-Neumann operator N and $\bar\partial N$ in Theorem 2 are new, even for domains with C^2-smooth boundaries in $\mathbb{C}^n$.

The $W^s(\Omega)$-regularity result for $\bar\partial N$ is sufficient and crucial in the proof of Theorem 1. The key observation is that the $\bar\partial_b$ equation (0.4), when restricted to each complex foliation leaf of M, is elliptic. Thus, any solution for (0.4) has Hölder regularity on each leaf from the elliptic theory. To prove continuity of the solution for equation (0.4), one only needs to prove continuity of the solution in the transversal direction, which can be proved by using the Besov norms and a finite difference scheme. Details are given in Section 5.

It should be pointed out that there exist *non-smooth* Levi-flat real hypersurfaces in $\mathbb{C}P^2$. Recall that a C^1 or Lipschitz real hypersurface M^{2n-1} is called Levi-flat in a complex manifold $\mathbb{C}P^n$ if $\mathbb{C}P^n \setminus M$ consists of two pseudoconvex domains. For example, let

$$M = \{[z_0, z_1, z_2] \in \mathbb{C}P^2 \mid |z_0| = |z_1|\},$$

where (z_0, z_1, z_2) are homogeneous coordinates in $\mathbb{C}P^2$. Then M is Levi-flat since $\mathbb{C}P^2 \setminus M$ consists of two pseudoconvex domains. The hypersurface M is smooth except at $(0, 0, 1)$, where M is neither C^1 nor Lipschitz (in the sense of a Lipschitz graph locally). It is still an open question if there exist Lipschitz or C^2 Levi-flat hypersurfaces in $\mathbb{C}P^n$.

In addition, the positive curvature condition (or the irreducible condition) is needed for the possible generalization of Theorem 1. One can inspect the example $S^1 \times \mathbb{C}P^1$, which is a C^∞-smooth Levi-flat hypersurface in $\mathbb{C}P^1 \times \mathbb{C}P^1$. Clearly, $\mathbb{C}P^1 \times \mathbb{C}P^1$ does not admit a Kähler metric with positive holomorphic bisectional curvature, (e.g., cf. [HSW]). Any compact Kähler manifold with positive holomorphic bisectional curvature must be biholomorphic to $\mathbb{C}P^n$ (e.g., cf. [Mok]). Although results in this paper are stated for domains in $\mathbb{C}P^n$, the conclusions of these results remain to be true for corresponding domains in any compact Kähler manifold with positive holomorphic bisectional curvature as well.

The plan of this paper is as follows: In Section 1 we derive some preliminary results on equidistant real hypersurfaces in $\mathbb{C}P^n$ for the proof of Theorems 1 and 2. In Section 2 we discuss the L^2 existence theorems of $\bar\partial$ and the $\bar\partial$-Neumann problem on domains in $\mathbb{C}P^n$. In Section 3 we prove that the $\bar\partial$-Neumann operator on pseudoconvex domains with C^2 boundary is regular in some Sobolev spaces as stated in Theorem 2. Using the existence and regularity of the $\bar\partial$-Neumann operator, one can study the solution of the $\bar\partial$-equation with prescribed support, including the case of forms with top degrees. We construct $\bar\partial$-closed extensions from the boundary and prove the existence and regularity of equation (0.4) in Section 4. The proof of Theorem 1 is completed in Section 5.

1. Equidistant hypersurfaces and preliminaries for Theorems 1-2

In this section, we first study the geometry of equidistant real hypersurfaces which will be needed to prove Theorem 1. Afterwards we recall four preliminary results which are indispensable in the proof of Theorem 2.

Suppose that Ω is a bounded domain with C^2-smooth boundary $b\Omega$ in $\mathbb{C}P^n$. Let $b\Omega_s = \{x \in \mathbb{C}P^n | d(x, b\Omega) = s\}$ and $\{b\Omega_s\}_{s<\epsilon}$ be a family of equidistant real hypersurfaces from $b\Omega$ in $\mathbb{C}P^n$.

We shall show that the connection and the curvature form of the complex line bundle of equidistant hypersurfaces $\{b\Omega_s\}_{s<\epsilon}$ are C^α, as long as $b\Omega$ is a $C^{2,\alpha}$-smooth hypersurface in $\mathbb{C}P^n$. We will use the Cartan-Chern theory to prove this assertion as follows.

Observe that, if $M = b\Omega$ is a C^2-smooth connected hypersurface in $\mathbb{C}P^n$, then M divides $\mathbb{C}P^n$ into two connected components Ω_+ and Ω_-. Let ρ be a C^2 defining function for M such that

$$\rho(x) = \begin{cases} -d(z, M), & z \in \Omega_-, \\ d(z, M), & z \in \Omega_+, \end{cases} \tag{1.1}$$

where $d(z, M)$ denotes the distance from z to M. The defining function ρ is called a signed distance function. Let $U_\epsilon(M) = \{z \in \mathbb{C}P^n \mid d(z, M) < \epsilon\}$ be a small tubular neighborhood of M. We consider the real two-dimensional normal plane bundle

$$\mathcal{N}_{\{\nabla\rho, J\nabla\rho\}} = \{(c_1\nabla\rho + c_2 J\nabla\rho)|_Q \mid Q \in U_\epsilon(M),\ c_1, c_2 \in \mathbb{R}\}$$

and the complex normal line bundle

$$\mathcal{N}_{\widetilde{\nabla\rho}} = \mathcal{N}_{\{\nabla\rho\otimes\mathbb{C}\}} = \{\lambda(\nabla\rho - \sqrt{-1}J(\nabla\rho))|_Q \mid Q \in U_\epsilon(M), \lambda \in \mathbb{C}\}.$$

The Levi-Civita connection D of $\mathbb{C}P^n$ with the standard Fubini-Study metric ω satisfies

$$D_\xi(J\eta) = J(D_\xi\eta) \tag{1.2}$$

since the metric ω is Kähler. We shall study the induced connections and curvature on $\mathcal{N}_{\{\nabla\rho, J\nabla\rho\}}$ and $\mathcal{N}_{\{\nabla\rho\otimes\mathbb{C}\}}$. There is also a hermitian connection $\tilde{D}$ on $\mathbb{C}T(\mathbb{C}P^n) = T^{1,0}(\mathbb{C}P^n) \oplus T^{0,1}(\mathbb{C}P^n)$. The two connections D and $\tilde{D}$ are related by

$$\tilde{D}_{\xi+\sqrt{-1}\eta}(Z + \sqrt{-1}W) = (D_\xi Z - D_\eta W) + \sqrt{-1}(D_\xi W + D_\eta Z)$$

for $\xi, \eta, Z, W \in [T(\mathbb{C}P^n)]_{\mathbb{R}}$.

Let us consider the corresponding real orthogonal decomposition

$$[T(\mathbb{C}P^n)]_{\mathbb{R}} = \mathcal{N}_{\{\nabla\rho, J\nabla\rho\}} \oplus \mathcal{N}^{\perp}_{\{\nabla\rho, J\nabla\rho\}}. \tag{1.3}$$

For $\xi \in [T(\mathbb{C}P^n)]_{\mathbb{R}}$, ξ^N is the projection of ξ into the normal component $\mathcal{N}_{\{\nabla\rho, J\nabla\rho\}}$. Similarly, we have

$$T^{1,0}(\mathbb{C}P^n) = \mathcal{N}_{\{\nabla\rho\otimes\mathbb{C}\}} \oplus \mathcal{N}^{\perp}_{\{\nabla\rho\otimes\mathbb{C}\}}. \tag{1.4}$$

For $\tilde{\xi} \in T^{1,0}(\mathbb{C}P^n)$, $\tilde{\xi}^N$ is the projection of $\tilde{\xi}$ into the complex normal component $\mathcal{N}_{\{\nabla\rho\otimes\mathbb{C}\}}$.

The map

$$u \mapsto \tilde{u} = \frac{1}{\sqrt{2}}(u - \sqrt{-1}Ju)$$

is a linear isomorphism from $T(\mathbb{C}P^n)_{\mathbb{R}}$ to $T^{1,0}(\mathbb{C}P^n)$. We define

$$D^N_\xi \eta = (D_\xi \eta)^N, \qquad \eta \in \mathcal{N}_{\{\nabla\rho, J(\nabla\rho)\}}. \tag{1.5}$$

The induced curvature tensor is defined by

$$R^N(Z,W)\eta = -D^N_Z D^N_W \eta + D^N_W D^N_Z \eta + D^N_{[Z,W]}\eta,$$

for $\eta \in N_{\{\nabla\rho, J(\nabla\rho)\}}$ and $Z, W \in [T(\mathbb{C}P^n)]_{\mathbb{R}}$. Similarly, we have the induced curvature tensor $\tilde{\Theta}^N$ for the complex line bundle $\mathcal{N}_{\widetilde{\nabla\rho}}$ on U_ϵ.

Since $\|\nabla\rho\|^2 = 1$,

$$d(\|\nabla\rho\|^2) = 2\langle D\nabla\rho, \nabla\rho\rangle = 0.$$

Thus $D^N_\xi(\nabla\rho)$ has no component in the $\nabla\rho$ direction and

$$D^N_\xi(\nabla\rho) = \beta(\xi)J(\nabla\rho) \tag{1.6}$$

where β on U_ϵ is defined by

$$\beta(\cdot) = \mathrm{Hess}(\rho)(\cdot, J(\nabla\rho)). \tag{1.7}$$

The 1-form β is the connection form for $\mathcal{N}_{\{\nabla\rho, J(\nabla\rho)\}}$. Recall that from (1.2), $D_\xi(J(\nabla\rho)) = JD_\xi(\nabla\rho)$ and $\langle\nabla\rho, J(\nabla\rho)\rangle \equiv 0$. Thus we have

$$D^N_\xi(J(\nabla\rho)) = JD^N_\xi(\nabla\rho) = J(\beta(\xi)J(\nabla\rho)) = -\beta(\xi)\nabla\rho. \tag{1.8}$$

We shall study the connection form for $\mathcal{N}_{\widetilde{\nabla\rho}}$ and its curvature form $\tilde{\Theta}^N$.

Proposition 1.1. *Let M be a C^m hypersurface in $\mathbb{C}P^n$ and let ρ be the signed distance function for M, $m \geq 2$. Let $\tilde{e}_n = \widetilde{\nabla\rho}$ be the unit complex normal on a tubular neighborhood $U_\epsilon(M)$ for sufficiently small $\epsilon > 0$. Then we have*

$$\tilde{D}_\xi \tilde{e}_n = \theta(\xi)\tilde{e}_n \tag{1.9}$$

where

$$\theta(\xi) = \sqrt{-1}Hess(\rho)(\xi, J(\nabla\rho)) = \sqrt{-1}\beta(\xi). \tag{1.10}$$

Consequently, $\sqrt{-1}\theta$ is real-valued and the curvature form $\tilde{\Theta}^N$ of $\mathcal{N}_{\widetilde{\nabla\rho}}$ is an exact 2-form on $U_\epsilon(M)$:

$$\tilde{\Theta}^N = -d\theta. \tag{1.11}$$

Furthermore, the connection θ and the curvature form $\tilde{\Theta}^N = -d\theta$ above are of class C^α if M is a $C^{2,\alpha}$-smooth hypersurface in $\mathbb{C}P^n$.

Proof. The complex line bundle $\mathcal{N}_{\widetilde{\nabla\rho}}$ is spanned by a global nowhere vanishing section $\widetilde{\nabla\rho} = \frac{1}{\sqrt{2}}(\nabla\rho - \sqrt{-1}J(\nabla\rho))$, the complex normal vector on $U_\epsilon(M)$. A direct computation shows that, by (1.6) and (1.8) we have

$$\begin{aligned}\tilde{D}^N_\xi \tilde{e}_n &= \frac{1}{\sqrt{2}}\{\tilde{D}^N_\xi \nabla\rho - \sqrt{-1}\tilde{D}^N_\xi(J(\nabla\rho))\} = \frac{1}{\sqrt{2}}\{\beta(\xi)J(\nabla\rho) + \sqrt{-1}\beta(\xi)(\nabla\rho)\}\\ &= \frac{\sqrt{-1}}{\sqrt{2}}\beta(\xi)\{(\nabla\rho) - \sqrt{-1}J(\nabla\rho)\} = \sqrt{-1}\beta(\xi)\tilde{e}_n = \theta(\xi)\tilde{e}_n,\end{aligned}$$

where $\theta = \sqrt{-1}\beta$ is the connection 1-form of the complex line bundle $N_{\widetilde{\nabla\rho}}$. This proves (1.9). Using the Chern formula [Chern], the curvature form $\tilde{\Theta}^N$ is given by

$$\tilde{\Theta}^N = -d\theta + \theta\wedge\theta = -d\theta = -\sqrt{-1}d\beta$$

which proves (1.11).

By (1.10), θ is $C^{m-2,\alpha}$ if M is $C^{m,\alpha}$. It remains to prove that the curvature form $\tilde{\Theta}^N$ of $N_{\widetilde{\nabla\rho}}$ is a 2-form with C^α coefficients on $U_\epsilon(M)$ if M is $C^{2,\alpha}$-smooth. This can be proved by the generalized Gauss-Codazzi equation (the Cartan-Chern-Gauss structure equation). Let $\tilde{e}_1, \cdots, \tilde{e}_n$ be an orthonormal local basis for $T^{1,0}(U_\epsilon \cap V)$ such that $\tilde{e}_n = \widetilde{\nabla\rho}$, where V is some neighborhood near a boundary point $p \in M$. Let $\tilde{\theta} = (\tilde{\theta}_{k\bar{l}})$ be the connection 1-form for $\mathbb{C}P^n$ defined by

$$\tilde{D}_\xi \tilde{e}_j = \sum_{l=1}^n \tilde{\theta}_{j,\bar{l}}\tilde{e}_l.$$

We remark that though $T^{1,0}(\mathbb{C}P^n)$ is a holomorphic vector bundle, our unitary frame $\{\tilde{e}_1, \cdots, \tilde{e}_n\}$ is not necessarily holomorphic. Thus $\tilde{\theta}_{k,\bar{l}}$ is not necessarily of $(1,0)$ type. Let $\tilde{R}(X,Y) = -\tilde{D}_X\tilde{D}_Y + \tilde{D}_Y\tilde{D}_X + \tilde{D}_{[X,Y]}$ and

$$\tilde{\Theta}_{k,\bar{l}}(Z,\overline{W}) = \langle \tilde{R}(Z,\overline{W})\tilde{e}_k, \bar{\tilde{e}}_l\rangle, \quad Z, W \in T^{1,0}(V). \tag{1.12}$$

The matrix-valued 2-tensor $\tilde{\Theta}$ is given by Chern's formula for the curvature of complex vector bundles (cf. [Chern], [KN] or [Zh]),

$$\tilde{\Theta} = -d\tilde{\theta} + \tilde{\theta}\wedge\tilde{\theta}. \tag{1.13}$$

In particular, we have

$$\begin{aligned}\tilde{\Theta}_{n,\bar{n}} &= \langle \tilde{R}(\cdot,\cdot)\tilde{e}_n, \bar{\tilde{e}}_n\rangle = -d\tilde{\theta}_{n,\bar{n}} + \sum_{n=1}^n \tilde{\theta}_{n,\bar{l}}\wedge\tilde{\theta}_{l,\bar{n}}\\ &= \tilde{\Theta}^N + \sum_{n=1}^{n-1}\tilde{\theta}_{n,\bar{l}}\wedge\tilde{\theta}_{l,\bar{n}},\end{aligned} \tag{1.14}$$

where we have used the fact that $\tilde{\theta}_{n,\bar{n}} = \theta$ and (1.11). It remains to calculate $\tilde{\theta}_{n,\bar{l}}$ and $\tilde{\theta}_{l,\bar{n}}$. For this purpose, we use the fact that

$$\langle \tilde{e}_n, \bar{\tilde{e}}_l \rangle = \langle \tilde{e}_l, \bar{\tilde{e}}_n \rangle = 0 \tag{1.15}$$

where $l = 1, \cdots, n-1$. Recall that for each $\xi \in \mathbb{C}T(V)$,

$$\tilde{D}_\xi \tilde{e}_n = \sum_{l=1}^{m} \tilde{\theta}_{n,\bar{l}}(\xi)\tilde{e}_l = \sum_{l=1}^{n} \langle \tilde{D}_\xi(\tilde{e}_n), \bar{\tilde{e}}_l \rangle \tilde{e}_l.$$

Thus, we have

$$\begin{aligned} \tilde{\theta}_{n,\bar{l}}(\xi) &= \langle \tilde{D}_\xi(\tilde{e}_n), \bar{\tilde{e}}_l \rangle = \frac{1}{2}\langle \tilde{D}_\xi(\nabla\rho - \sqrt{-1}J(\nabla\rho)), e_l + \sqrt{-1}Je_l \rangle \\ &= \frac{1}{2}\langle \tilde{D}_\xi(\nabla\rho), e_l + \sqrt{-1}Je_l \rangle \ - \frac{1}{2}\langle \sqrt{-1}J\tilde{D}_\xi(\nabla\rho), e_l + \sqrt{-1}Je_l \rangle \\ &= \langle \tilde{D}_\xi(\nabla\rho), e_l \rangle + \sqrt{-1}\langle \tilde{D}_\xi(\nabla\rho), Je_l \rangle = \sqrt{2}\mathrm{Hess}(\rho)(\xi, \tilde{e}_l). \end{aligned} \tag{1.16}$$

Similarly, we have

$$\begin{aligned} \tilde{\theta}_{l,\bar{n}}(\xi) &= \langle \tilde{D}_\xi(\tilde{e}_l), \bar{\tilde{e}}_n \rangle = -\langle \tilde{e}_l, \tilde{D}_\xi(\bar{\tilde{e}}_n) \rangle \\ &= -\langle \tilde{D}_{\bar{\xi}}(\tilde{e}_n), \bar{\tilde{e}}_l \rangle = -\sqrt{2}\mathrm{Hess}(\rho)(\bar{\xi}, \tilde{e}_l). \end{aligned} \tag{1.17}$$

This implies that

$$\tilde{\theta}_{l,\bar{n}} = -\bar{\tilde{\theta}}_{n,\bar{l}}, \qquad \text{on } V. \tag{1.18}$$

Thus

$$\tilde{\theta}_{n,\bar{l}} \wedge \tilde{\theta}_{l,\bar{n}} = -\tilde{\theta}_{n,\bar{l}} \wedge \bar{\tilde{\theta}}_{n,\bar{l}} \tag{1.19}$$

is a skew-hermitian 2-form.

If ρ is $C^{m,\alpha}$-smooth, then each $\tilde{e}_k$ is $C^{m-1,\alpha}$-smooth for $k = 1, \cdots n$. Thus $\tilde{\Theta}_{n,\bar{n}}$ is $C^{m-1,\alpha}$-smooth. It follows from (1.14) and (1.16)-(1.18) that $\tilde{\Theta}^N$ is a $C^{m-2,\alpha}$ smooth form, since $\mathrm{Hess}(\rho)$ is $C^{m-2,\alpha}$. The proposition is proved. $\square$

The C^α-regularity of θ and $\tilde{\Theta}^N$ will be used in the proof of Theorem 1, see Section 5 below. When $M = b\Omega$ is *a Levi-flat real-hypersurface*, the restriction Θ_b on M of the curvature form $\tilde{\Theta}^N$ above has some additional properties.

Proposition 1.2. *Suppose that M is a C^2-smooth Levi-flat hypersurface in $\mathbb{C}P^n$ with the signed distance function ρ. Let $\widetilde{\nabla\rho}$ be the unit complex normal on a tubular neighborhood $U_\epsilon(M)$ for sufficiently small $\epsilon > 0$ and let $\tilde{\Theta}^N$ be the curvature tensor for the complex line bundle $N_{\widetilde{\nabla\rho}}$. If J is the complex structure of $\mathbb{C}P^n$, then $\Theta_b = \tilde{\Theta}^N|_{[T(M)]_{\mathbb{R}} \cap J[T(M)]_{\mathbb{R}}}$ is a $(1,1)$-form. Furthermore, we have*

$$\sqrt{-1}\tilde{\Theta}^N(\tau, \bar{\tau}) = \sqrt{-1}\Theta_b(\tau, \bar{\tau}) \geq 2 \tag{1.20}$$

where $\tau \in T^{1,0}(M)$ with $|\tau| = 1$.

Proof. If M^{2n-1} is a Levi-flat real hypersurface, then by definition we have

$$\mathrm{Hess}(\rho)(\bar{\tilde{Y}}, \tilde{X}) = (i\partial\bar{\partial}\rho)(\bar{\tilde{Y}}, \tilde{X}) = 0 \tag{1.21}$$

for any pair $\{\tilde{X}, \tilde{Y}\} \in T^{(1,0)}(M)$.

It follows from (1.16)-(1.17) and (1.21) that the connection forms $\{\tilde{\theta}_{n,\bar{l}}\}$ are $(1,0)$-forms on the Levi-flat hypersurface $M = b\Omega$. Using (1.14) and (1.18), we have

$$\tilde{\Theta}^N = \tilde{\Theta}_{n,\bar{n}} - \sum_{n=1}^{n-1} \tilde{\theta}_{n,\bar{l}} \wedge \tilde{\theta}_{l,\bar{n}} = \tilde{\Theta}_{n,\bar{n}} + \sum_{n=1}^{n-1} \tilde{\theta}_{n,\bar{l}} \wedge \bar{\tilde{\theta}}_{n,\bar{l}}. \tag{1.22}$$

Because the Fubini-Study metric ω is a Kähler metric, its curvature form $\tilde{\Theta}$ is a matrix valued (1, 1)-form with respect to any unitary frame (cf. [Chern], [Zh]). Hence, $\tilde{\Theta}_{n,\bar{n}}$ is a (1, 1)-form. Since $\{\tilde{\theta}_{n,\bar{l}}\}$ are $(1,0)$-forms on the Levi-flat hypersurface $M = b\Omega$, we see that $\sqrt{-1}\sum_{n=1}^{n-1} \tilde{\theta}_{n,\bar{l}} \wedge \bar{\tilde{\theta}}_{n,\bar{l}}$ is a nonnegative $(1,1)$-form. This fact together with (1.22) implies that Θ_b is a $(1,1)$-form. Furthermore, it follows that, for $\tau \in T^{(1,0)}(M)$ with $|\tau| = 1$,

$$\sqrt{-1}\tilde{\Theta}^N(\tau, \bar{\tau}) \geq \sqrt{-1}\tilde{\Theta}_{n,\bar{n}}(\tau, \bar{\tau}) \geq 2,$$

where we used the fact that the Fubini-Study metric of $\mathbb{C}P^n$ has positive holomorphic bisectional curvature ≥ 2, (e.g., cf. [KN], [Zh]). This completes the proof. □

Let us now recall some preliminary results, which play important roles in the proof of Theorem 2. First of all, we need to recall the so-called $\bar{\partial}$-Neumann boundary condition. The L^2-theory of the $\bar{\partial}$-equation $\bar{\partial}u = v$ and the Laplace equation $\Box u = f$ are related to the formal adjoint ϑ of $\bar{\partial}$ on the space $L^2_{(p,q)}(\Omega)$. Let $*$ be the real Hodge star operator of the Riemannian metric and let $*$ be extended $\mathbb{C}$-linearly on the space of (p,q)-forms. Then

$$\vartheta u = -\bar{*}[\bar{\partial}(\bar{*}u)] = - * \partial * u. \tag{1.23}$$

For $u, v \in L^2(\Omega)$, we use

$$(u, \bar{v})_{L^2(\Omega)} = \int_\Omega \langle u, \bar{v} \rangle$$

to denote the inner product. For $q > 0$, using integration by parts on the domain Ω with C^2-smooth boundary $b\Omega$, we have that $u \in \mathrm{Dom}(\bar{\partial}^*|_\Omega) \cap C^1_{(p,q)}(\overline{\Omega})$ if and only if

$$(\vartheta u, \bar{v})_{L^2(\Omega)} = (\bar{\partial}^* u, \bar{v})_{L^2(\Omega)} = (u, \overline{\bar{\partial} v})_{L^2(\Omega)} \tag{1.24}$$

holds for all $v \in L^2_{(p,q)}(\Omega)$. The condition (1.24) is equivalent to the so-called $\bar{\partial}$-Neumann boundary condition

$$u \llcorner_{(\bar{\partial}\rho)_\#} = 0 \text{ on } b\Omega, \tag{1.25.1}$$

where $(\bar{\partial}\rho)_\# = \frac{1}{\sqrt{2}}(\nabla \rho + \sqrt{-1} J \nabla \rho)$ is the dual of $\bar{\partial}\rho$. In general, $u \in \mathrm{Dom}(\bar{\partial}^*|_\Omega)$ implies that (1.25.1) holds as currents.

Throughout this paper, we let

$$\mathrm{Dom}(\bar{\partial}^*) = \mathrm{Dom}(\bar{\partial}^*|_\Omega) = \{u \in L^2_{(p,q)}(\Omega) | u \llcorner_{(\bar{\partial}\rho)_\#} = 0 \text{ on } b\Omega\} \tag{1.25.2}$$

and

$$\mathrm{Dom}(\Box|_\Omega) = \{u \in L^2_{(p,q)}(\Omega) | u \llcorner_{(\bar{\partial}\rho)_\#} = 0 \text{ and } (\bar{\partial} u) \llcorner_{(\bar{\partial}\rho)_\#} = 0 \text{ on } b\Omega\}. \tag{1.25.3}$$

There always exists the *weak* Hodge-Kodaira decomposition for any domain Ω:

$$L^2_{(p,q)}(\Omega) = \ker(\Box) \oplus_{L^2} \overline{\mathrm{Range}(\Box|_\Omega)}, \tag{1.26}$$

where $\overline{\mathrm{Range}(\Box|_\Omega)}$ denotes the closure of $\mathrm{Range}(\Box|_\Omega)$ in $L^2(\Omega)$. A necessary condition for the existence of the $\bar{\partial}$-Neumann operator on $L^2_{(p,q)}(\Omega)$ is the condition that the range of $\Box$ must be closed, i.e., $\overline{\mathrm{Range}(\Box)} = \mathrm{Range}(\Box)$. To prove the first part of Theorem 2, by the definition of $\Box = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$, it is sufficient to show that both $\mathrm{Range}(\bar{\partial})$ and $\mathrm{Range}(\bar{\partial}^*)$ are closed subspaces of $L^2_{(p,q)}(\Omega)$. For this purpose (and for the convenience of the reader), we recall the following elementary but very useful fact in functional analysis (cf. [Hö2-3] or [CS, p60]):

Lemma 1.3. *Let $\mathcal{F} : H_1 \to H_2$ be a linear, closed, densely defined operator between two Hilbert spaces. The following conditions on $\mathcal{F}$ are equivalent:*

(1) Range($\mathcal{F}$) is closed;

(2) There is a constant C such that

$$\|u\|_1 \le C \|\mathcal{F} u\|_2 \text{ for all } u \in Dom(\mathcal{F}) \cap \overline{Range(\mathcal{F}^*)}; \tag{1.27.1}$$

(3) Range($\mathcal{F}^*$) is closed;

(4) There is a constant C such that

$$\|v\|_2 \leq C\|\mathcal{F}^* v\|_2 \text{ for all } v \in Dom(\mathcal{F}^*) \cap \overline{Range(\mathcal{F})}; \tag{1.27.2}$$

The best constants in (1.27.1)-(1.27.2) are the same.

Lemma 1.3 will be used in the proofs of Theorem 2.1, Theorem 2.6 and Theorems 3.4-3.5 below, which are parts of Theorem 2. By Lemma 1.3, it suffices to derive *a priori* estimates for the $\bar{\partial}$ equation and the $\Box$ equation. We recall the curved version of the Morrey-Kohn-Hörmander formula, which is a Bochner type formula with a weight function $e^{-\phi}$. The weighted function $e^{-\phi}$ induces a perturbed adjoint operator:

$$\vartheta_\phi u = \bar{\partial}^*_\phi u = e^{\phi}\vartheta(e^{-\phi u}) = -e^{\phi} * \partial(e^{-\phi} * u). \tag{1.28}$$

Of course, one can formulate the weighted Laplace operator $\Box_\phi = (\bar{\partial}\bar{\partial}^*_\phi + \bar{\partial}^*_\phi\bar{\partial})$ and the corresponding weighted $\bar{\partial}$-Neumann operators N_ϕ.

Our strategy is to use the weighted $\bar{\partial}$-Neumann operators N_ϕ to estimate the original $\bar{\partial}$-Neumann operator N, see Sections 2-3 below. In order to estimate the operator $\bar{\partial}^*_\phi N_\phi$, we let $L_1, \cdots, L_n$ be a local orthonormal frame for $T^{(1,0)}(U)$, where U is a local neighborhood near some point in $\overline{\Omega}$. For any $u \in C_{(p,q)}(\overline{\Omega})$ and $\phi \in C^2(\Omega)$, we define an operator

$$\langle (i\partial\bar{\partial}\phi)u, \overline{u}\rangle = \sum_{j,k=1}^{n} (i\partial\bar{\partial}\phi)(L_j, \overline{L}_k)\langle u \llcorner \overline{L}_j, \overline{u}\llcorner L_k\rangle \text{ on } \Omega. \tag{1.29}$$

Notice that $\langle (i\partial\bar{\partial}\phi)u, \overline{u}\rangle$ is independent of the choice of a local unitary basis.

Similarly, for $Q \in b\Omega$, we require that $L_n = \sqrt{2}(\partial\rho)_\#$ and define

$$\langle (i\partial\bar{\partial}\rho)u, \overline{u}\rangle = \sum_{j,k=1}^{n-1} (i\partial\bar{\partial}\rho)(L_j, \overline{L}_k)\langle u \llcorner \overline{L}_j, \overline{u}\llcorner L_k\rangle \text{ on } b\Omega. \tag{1.30}$$

In addition, since $\mathbb{C}P^n$ has non-zero curvature, we set

$$\langle \Theta u, \overline{u}\rangle = \sum_{j,k}^{n} \langle \overline{w}^j \wedge \{[R(L_j, \overline{L}_k)u]\llcorner \overline{L}_k\}, \overline{u}\rangle \text{ on } \Omega, \tag{1.31}$$

where $\{w^j\}$ is the dual of the frame $\{L_k\}$ and $R_{X,Y} = -D_X D_Y u + D_Y D_X + D_{[X,Y]}$ is the curvature operator on (p,q)-forms, see [Wu, Chapter 2]. Finally, we set

$$(u, \overline{v})_\phi = \int_\Omega \langle u, \overline{v}\rangle e^{-\phi}. \tag{1.32}$$

Proposition 1.4. *(Bochner-Hörmander-Kohn-Morrey formula) Let Ω be a compact domain Ω with C^2-smooth boundary $b\Omega$ and $\rho(x) = -d(x, b\Omega)$. Then, for any (p,q)-form $u \in Dom(\bar\partial) \cap Dom(\bar\partial^*)$ with $q \geq 1$ on Ω, we have*

$$(1.33)\quad \|\bar\partial u\|^2_\phi + \|\bar\partial^*_\phi u\|^2_\phi = \|\overline{\nabla} u\|^2_\phi + (\Theta u, \overline{u})_\phi + ((i\partial\bar\partial\phi)u, \overline{u})_\phi + \int_{b\Omega} \langle (i\partial\bar\partial\rho)u, \overline{u}\rangle e^{-\phi},$$

where $\|\overline{\nabla} u\|^2_\phi = \int_\Omega \sum_{j=1}^n |D_{\bar L_j} u|^2 e^{-\phi}$ and $\{L_1, ..., L_n\}$ is a local unitary frame of $T^{(1,0)}(\Omega)$.

Proof. This formula is known (cf. [AV], [Hö2] [Dem1-2], [Siu1], [Wu]) for some special cases, although it has not been stated in the literature in the form (1.33). If u has compact support in the interior of Ω, the formula (1.33) was proved in [AV], Chapter 8 of [Dem2] and (2.12) of [Wu].

It remains to discuss the boundary term of (1.33). For the case $\phi = 0$, the stated formula with the boundary term was proved in [Siu1]. To compute the boundary term, one notice that although the boundary terms in the proof of (1.33) involve the weight function, they have nothing to do with the Riemannian curvature R nor Θ. The boundary term had been computed in Hörmander [Hö2, Chapter 3] by combining the Morrey-Kohn technique on the boundary with non-trivial weight function. If one combines the results of [Hö2] with the interior formulae discussed above, one can prove that (1.33) holds for the general case with a weight function $e^{-\phi}$ and the curvature term. □

In later application, we will choose $\phi = -t \log|\rho|$ in (1.33). Therefore, we need to discuss the term $i\partial\bar\partial\phi$ with $\phi = -t\log|\rho|$ and the curvature term $\langle \Theta u, \bar u\rangle$ in the right hand side of (1.33).

Proposition 1.5. *Let u be a (p, q)-form of $\Omega \subset \mathbb{C}P^n$ with $q \geq 1$. Suppose that Θ is the curvature term defined in (1.31) with respect to the Fubini-Study metric. Then we have*

$$(1.34)\qquad \begin{cases} \langle \Theta u, \overline{u}\rangle & = 0, \quad \text{for any } (n,q)\text{-form } u; \\ \langle \Theta u, \overline{u}\rangle & \geq 0, \quad \text{when } p \geq 1 \text{ and } u \text{ is a } (p,\ q)\text{-form}; \\ \langle \Theta u, \overline{u}\rangle & = q(2n+1)|u|^2, \quad \text{when } u \text{ is a } (0,\ q)\text{-form.} \end{cases}$$

Proof. The assertion for $(0, q)$-forms and (n, q)-forms was computed in [Wu] and [Siu1]. For the curvature operator Θ acting on (1, 1)-forms u, [Siu1] showed that Θ is non-negative definite, but *not* positive definite, (see [Pe] as well). Henkin and Iordan [HI] extended this observation for all (p, q)-forms with $p \geq 1$. In fact, Lemma 3.3 of [HI] and its proof showed that Θ acting on $L^2_{(p,q)}(\Omega)$ is a non-negative operator. Furthermore, Θ is *positive definite* on $L^2_{(p,q)}(\Omega)$ if and only if $p = 0$ and $q \geq 1$. □

Proposition 1.6. *If Ω is a pseudoconvex domain with C^2-smooth boundary $b\Omega$ in $\mathbb{C}P^n$ and ρ is the signed distance function, then*

$$i\partial\bar{\partial}(-\log|\rho|)(X,\overline{X}) \geq \frac{1}{2}|X|^2 \tag{1.35}$$

for any $X \in T^{(1,0)}(\Omega)$.

Proposition 1.6 was proved in Takeuchi (e.g., cf. [Su], [Ta]). A new proof of Proposition 1.6 using comparison theorems in Riemannian geometry will appear in another paper in [CaS].

2. L^2 theory for $\bar{\partial}$ on pseudoconvex domains in $\mathbb{C}P^n$

Let Ω be a bounded pseudoconvex domain in $\mathbb{C}P^n$. We fix our Fubini-Study metric ω for $\mathbb{C}P^n$. In this section, we will use preliminary results of Section 1 to study the $\bar{\partial}$-Neumann operators N, $\bar{\partial}^*N$ and $\bar{\partial}N$ acting on $L^2_{(p,q)}(\Omega)$. We first prove the L^2 existence theorem of the $\bar{\partial}$-Neumann operator for the easier case when $p=0$.

Theorem 2.1. *Let Ω be a pseudoconvex domain in $\mathbb{C}P^n$, $n \geq 2$. Then the $\bar{\partial}$-Neumann operator $N_{(0,q)}$ exists and is bounded on $L^2_{(0,q)}(\Omega)$, where $0 \leq q \leq n$. For any $f \in L^2_{(0,q)}(\Omega)$,*

$$\begin{aligned} f &= \bar{\partial}\bar{\partial}^* N_{(0,q)} f \oplus \bar{\partial}^*\bar{\partial} N_{(0,q)} f, \quad 1 \leq q \leq n-1. \\ f &= \bar{\partial}^*\bar{\partial} N_{(0,0)} f \oplus Pf, \quad q = 0, \end{aligned} \tag{2.1}$$

where $P: L^2(\Omega) \to L^2(\Omega) \cap Ker(\bar{\partial})$ is the Bergman projection with

$$P = I - \bar{\partial}^* N_{(0,1)} \bar{\partial} \tag{2.2}$$

and

$$N_{(0,0)} = \vartheta N^2_{(0,1)} \bar{\partial}. \tag{2.3}$$

Moreover, N, $\bar{\partial}^*N$ and $\bar{\partial}N$ are bounded operators with respect to the L^2-norms.

Proof. We first assume that Ω is a pseudoconvex domain in $\mathbb{C}P^n$ with C^2 boundary M. Let $1 \leq q \leq n$ and let ρ be a C^2 defining function for Ω normalized by $|d\rho| = 1$ on M. Choosing $\phi = 0$ in Proposition 1.4 , we have for any $(0,q)$-form $u \in \mathrm{Dom}(\bar{\partial}) \cap \mathrm{Dom}(\bar{\partial}^*)$,

$$\|\bar{\partial}u\|^2 + \|\bar{\partial}^*u\|^2 = \|\overline{\nabla}u\|^2 + \int_M \langle (i\partial\bar{\partial}\rho)u, \bar{u}\rangle + (\Theta u, \bar{u}) \geq \|\overline{\nabla}u\|^2 + (\Theta u, \bar{u}), \tag{2.4.1}$$

where Θ is the Ricci form and $\overline{\nabla}$ is the holomorphic gradient. Since the Ricci curvature of $\mathbb{C}P^n$ with the Fubini-Study metric is equal to 2n + 1, (cf. [KN] [Pe] or [Zh]), by (2.4.1) we have

$$(2.4.2) \qquad Q(u,\bar{u}) = \|\bar{\partial}u\|^2 + \|\bar{\partial}^* u\|^2 \geq q(2n+1)\|u\|^2, \quad u \in \mathrm{Dom}(\bar{\partial}) \cap \mathrm{Dom}(\bar{\partial}^*).$$

Consequently, we have

$$(2.4.3) \qquad (\Box u, \bar{u}) = Q(u,\bar{u}) \geq q(2n+1)\|u\|^2, \quad u \in \mathrm{Dom}(\Box_{(0,q)}|_\Omega),$$

for $q \geq 1$.

It follows from (2.4.3) that $\mathrm{Ker}(\Box_{(0,q)}) = 0$ and $\|N_{(0,q)}\|_{L^2} \leq \frac{1}{q(2n+1)}$ for $q \geq 1$. Moreover, by (2.4.2)-(2.4.3) and Lemma 1.3, one can show that the three operators $\bar{\partial}, \bar{\partial}^*$ and $\Box$ have closed ranges in $L^2(\Omega)$. Notice that if $\bar{\partial} f = 0$ and if $u = \bar{\partial}^* N f$, then u is the solution of $\bar{\partial}u = f$ with the smallest L^2-norm. Using this fact, by (2.4.2) and Lemma 1.3, one can further show that $\bar{\partial}^* N$ is a bounded operator. Since $\bar{\partial}N = (N\bar{\partial}^*)^* = (\bar{\partial}^* N)^*$, using Lemma 1.3 again, we see that the operator $\bar{\partial}N$ is bounded as well. This proves Theorem 2.1 for the case of $q \geq 1$.

The existence of $N_{(0,0)}$ also follows and one can prove (2.2) and (2.3) exactly as for domains in $\mathbb{C}^n$. The general case for nonsmooth domains follows from exhausting a pseudoconvex domain by C^2 pseudoconvex domains and using (2.4) (See proofs of Theorem 4.4.1 and Theorem 4.4.3 in [CS] for details). $\Box$

When $0 < p \leq n$, the proof of the L^2 existence of the $\bar{\partial}$-Neumann operator is more involved. we will use Lemma 1.3 and Propositions 1.4-1.6 with non-trivial weight function $e^{-\phi} = |\rho|^t$. Here is a preliminary result on the complex Hessian of the weight function.

Lemma 2.2. *Let $\Omega \subset\subset \mathbb{C}P^n$ be a pseudoconvex domain with C^2-smooth boundary $b\Omega$ and let $\delta(x) = d(x, b\Omega)$ be the distance function to $b\Omega$. Let $t_0 = t_0(\Omega)$ be the order of plurisubharmonicity for the distance function δ defined by (0.7). Then, for any $0 < t < t_0$, there exist $C_t > 0$ such that*

$$(2.5) \qquad i\partial\bar{\partial}(-\delta^t) \geq C_t \delta^t \left(\omega + \frac{i\partial\delta \wedge \bar{\partial}\delta}{\delta^2}\right),$$

where ω is the Kähler form of the Fubini-Study metric on $\mathbb{C}P^n$.

Proof. The existence of such t_0 for pseudoconvex domains in $\mathbb{C}^n$ with C^2-smooth boundary in $\mathbb{C}P^n$ is proved in Ohsawa-Sibony [OS]. Thus, by the inequality $i\partial\bar{\partial}(-\delta^{t_0}) \geq 0$ with $0 < t_0 \leq 1$, we obtain that

$$(2.6) \qquad i\frac{\partial\bar{\partial}(-\delta)}{\delta} + (1-t_0)\frac{i\partial\delta \wedge \bar{\partial}\delta}{\delta^2} \geq 0$$

Using Proposition 1.6, we have

$$i\partial\bar{\partial}(-\log\delta) = i\frac{\partial\bar{\partial}(-\delta)}{\delta} + \frac{i\partial\delta\wedge\bar{\partial}\delta}{\delta^2} \geq C\omega \tag{2.7}$$

where $C = \frac{1}{2}$. Multiplying (2.6) by $(1-\epsilon)$ and multiplying (2.7) by ϵ, and adding the two inequalities together (i.e., using the sum $(1-\epsilon)(2.6)+\epsilon(2.7)$), we conclude that, for any $0 \leq \epsilon \leq 1$, the inequality

$$i\partial\bar{\partial}(-\log\delta) = i\frac{\partial\bar{\partial}(-\delta)}{\delta} + \frac{i\partial\delta\wedge\bar{\partial}\delta}{\delta^2} \geq C\epsilon\omega + (1-\epsilon)t_0\frac{i\partial\delta\wedge\bar{\partial}\delta}{\delta^2} \tag{2.8}$$

holds. Hence, for any $0 < t < t_0$, we choose ϵ_t such that $(1-\epsilon_t)t_0 > t$. Then

$$\begin{aligned} i\partial\bar{\partial}(-\delta^t) &= it\delta^t\left(\frac{\partial\bar{\partial}(-\delta)}{\delta} + (1-t)\frac{\partial\delta\wedge\bar{\partial}\delta}{\delta^2}\right) \\ &\geq C_t t\delta^t\left(\omega + \frac{i\partial\delta\wedge\bar{\partial}\delta}{\delta^2}\right) \end{aligned}$$

where $C_t = \min(\frac{1}{2}\epsilon_t, (1-\epsilon_t)t_0 - t)$. The lemma is proved. $\square$

We will use Hörmander's weight function method to obtain the L^2 existence for the $\bar{\partial}$-equation with weights first. Let t be any real number and $\phi \in C^2(\Omega)$. Let $L^2(\delta^t)$ denote the L^2 space with respect to the weight function $e^{-t\phi} = \delta^t$ and

$$||f||^2_{(t)} = \int_\Omega |f|^2 e^{-t\phi} = \int_\Omega \delta^t|f|^2.$$

We use $\bar{\partial}^*_t$ to denote the adjoint of $\bar{\partial}$ with respect to the weighted space.

We use the norm $||.||^2_{(t)}$ since it is equivalent to the Sobolev norm on a sub-space of $W^{-\frac{t}{2}}(\Omega)$, (see [CS p348] and [MS1-2]). In fact, we have

$$||\tilde{u}||^2_{(t)} \simeq ||\tilde{u}||_{W^{-\frac{t}{2}}(\Omega)} \tag{2.9}$$

when $\tilde{u}$ satisfies an elliptic equation such as $\bar{\partial}\oplus\vartheta$.

In order to use Proposition 1.4 to derive the desired estimates for the operator $\bar{\partial}^*N$, we need to introduce the following two asymmetric weighted norms. These new norms will be used to obtain more refined estimates than those obtained in [Hö2].

For any (p,q)-form f on Ω, we decompose f into complex normal and tangential parts by setting

$$\begin{cases} f^\nu &= (f\llcorner_{(\bar{\partial}\delta)_\#})\wedge\bar{\partial}\delta \\ f^\tau &= f - f^\nu. \end{cases}$$

The above decomposition is well-defined for any (p,q)-form f supported in $[\Omega \setminus \mathrm{Cut}_\Omega(b\Omega)]$, where $\mathrm{Cut}_\Omega(b\Omega)$ is the cut-loci of $b\Omega$ in Ω, see [Cha], [CE] or [Pe]. It is well-known that the cut-loci $\mathrm{Cut}_\Omega(b\Omega)$ of $b\Omega$ has real dimension $\leq (2n-1)$ and hence has zero measure in Ω, (e.g., [Cha], [CE, p90ff], [Pe]). In summary, the above decomposition exists *almost everywhere in* Ω.

We define the asymmetric weighted norm

$$|f|_A^2 = |f^\tau|^2 + \frac{|f^\nu|^2}{|\delta|^2}. \tag{2.10.1}$$

We also define the dual norm

$$|f|_{A'}^2 = |f^\tau|^2 + |f^\nu|^2|\delta|^2. \tag{2.10.2}$$

For any $t > 0$, let $L^2_A(\delta^t)$ and $L^2_{A'}(\delta^t)$ denote the weighted L^2 spaces on (p,q)-forms defined by the norm

$$|||u|||^2_{(t)} = \int_\Omega \delta^t |f|_A^2 = \int_\Omega \delta^t (|f^\tau|^2 + \frac{|f^\nu|^2}{|\delta|^2}) \tag{2.10.3}$$

and

$$|||u|||^2_{(t)'} = \int_\Omega \delta^t |f|_{A'}^2 = \int_\Omega \delta^t |(f^\tau|^2 + |f^\nu|^2|\delta|^2). \tag{2.10.4}$$

We may assume that $\mathrm{Diam}(\mathbb{C}P^n) \leq 1$ up to a factor $\frac{\pi}{2}$. It is obvious that

$$|||u|||^2_{(t)'} \leq \| u \|^2_{(t)} \leq |||u|||^2_{(t)},$$

if $\mathrm{Diam}(\Omega) \leq 1$.

The following is a preliminary estimate for the operator $\bar\partial^* N$, because $u = \bar\partial^* N f$ is the solution of $\bar\partial u = f$ with the least L^2-norm.

Proposition 2.3. *Let $\Omega \subset\subset \mathbb{C}P^n$ be a pseudoconvex domain with C^2-smooth boundary $b\Omega$. Let $t_0 = t_0(\Omega)$ be the order of plurisubharmonicity for the distance function δ defined by (0.7). For any $0 < t < t_0$ and any (p,q)-form $f \in L^2_{A'}(\delta^t)$, where $0 \leq p \leq n$ and $1 \leq q \leq n$, such that $\bar\partial f = 0$ in Ω, there exists $u \in L^2_{(p,q-1)}(\delta^t)$ satisfying $\bar\partial u = f$ and*

$$\|u\|^2_{(t)} \leq \frac{1}{C_0 t} |||f|||^2_{(t)'}. \tag{2.11}$$

Proof. By Proposition 1.6, we have that $\phi = -\log\delta$ is strictly plurisubharmonic and $i\partial\bar\partial\phi \geq \frac{1}{2}\omega$, where ω is the Kähler form of $\mathbb{C}P^n$ with the Fubini-Study metric.

It is known that $\mathrm{Dom}(\bar\partial_t^*) = \mathrm{Dom}(\bar\partial^*)$, (e.g., [CS, Chapter 4]). Using Hörmander's weighted estimates (cf. Proposition 1.4) and Proposition 1.5, we have the following formula: for any (p,q)-form $g \in \mathrm{Dom}(\bar\partial) \cap \mathrm{Dom}(\bar\partial_t^*)$,

$$\|\bar\partial g\|^2_{(t)} + \|\bar\partial_t^* g\|^2_{(t)} \geq t((i\partial\bar\partial)\phi g, \bar g)_{(t)}. \tag{2.12.1}$$

From (2.8), the positive (1,1)-form $i\partial\bar\partial\phi$ induces a pointwise norm on (p,q) forms

$$|g|^2_{i\partial\bar\partial\phi} = \langle i(\partial\bar\partial\phi)g, \bar g\rangle \geq C_0|g|^2_A, \tag{2.12.2}$$

where $C_0 = \min\left(\frac{1}{2}, t_0\right)$ with $t_0 = t_0(\Omega)$ as in (2.5). Thus, we have

$$\|\bar\partial g\|^2_{(t)} + \|\bar\partial_t^* g\|^2_{(t)} \geq C_0 t|||g|||_{(t)}. \tag{2.13}$$

Let $(i\partial\bar\partial\phi)'$ denote the dual norm for (p,q)-forms induced by $i\partial\bar\partial\phi$. Using (2.13) and the same argument as in Demailly [Dem1,2], we conclude that for any $f \in L^2_{A'}(\delta^t)$, there exist $u \in L^2(\delta^t)$ satisfying $\bar\partial u = f$ and

$$\int_\Omega |u|^2\delta^t \leq \int_\Omega |\bar\partial u|^2_{(i\partial\bar\partial\phi)'}\delta^t \leq \frac{1}{C_0 t}\int_\Omega |\bar\partial u|^2_{A'}\delta^t. \tag{2.14}$$

This completes the proof. □

Notice that the terms in (2.14) have an extra weight factor δ^t. We need to remove this factor δ^t in order to obtain the desired L^2-estimate for all $p \geq 0$. The next two propositions have already been obtained in Berndtsson-Charpentier [BC] (see also [HI]). We include the proof here for the sake of completeness.

Proposition 2.4. *Let $\Omega \subset\subset \mathbb{C}P^n$ be a pseudoconvex domain with C^2 boundary $b\Omega$. For any $f \in L^2_{(p,q)}(\Omega)$, where $0 \leq p \leq n$ and $1 \leq q \leq n$, such that $\bar\partial f = 0$ in Ω, there exists $u \in L^2_{(p,q-1)}(\Omega)$ satisfying $\bar\partial u = f$ with $\int_\Omega |u|^2 \leq \hat C \int_\Omega |f|^2$.*

Proof. By Proposition 2.3, for any $t > 0$ with $0 < t < t_0/4$, there exists $u \in L^2_{(p,q-1)}(\delta^t)$ satisfying $\bar\partial u = f$, such that u is perpendicular to $\mathrm{Ker}(\bar\partial)$ in $L^2(\delta^t)$ and u satisfies inequality (2.11).

Consider $v = u\delta^{-t}$. Then $v \in L^2(\delta^{2t})$ and $v \perp \mathrm{Ker}(\bar\partial)$ in $L^2(\delta^{2t})$. It follows from (2.14) that the following holds:

$$\int_\Omega |u|^2 = \int_\Omega |v|^2\delta^{2t} \leq \frac{1}{2C_0 t}\int_\Omega |\bar\partial v|^2_{A'}\delta^{2t}. \tag{2.15}$$

By the definitions of the norm $|.|_{A'}$ and $\phi = -\log\delta$ (cf. (2.10.2)), we have

$$\begin{aligned} |\bar\partial v|^2_{A'} e^{-2t\phi} &\leq 2|\bar\partial u|^2_{A'} + 2t^2|\bar\partial\phi|^2_{A'}|u|^2 \\ &\leq C(|\bar\partial u|^2 + 2t^2|u|^2). \end{aligned} \tag{2.16}$$

Choosing t sufficiently small and substituting (2.16) into (2.15), one obtains

$$\int_\Omega |u|^2 \leq \hat{C}\int_\Omega |\bar\partial u|^2, \tag{2.17}$$

where $\hat{C} = \frac{C}{2C_0t(1-2Ct^2)}$. This proves the theorem. □

By (2.9), the Sobolev norm $\|u\|_{W^{\frac{t}{2}}_{(p,q)}(\Omega)}$ of the solution u is related to the weighted norm $\|u\|_{(-t)}$.

Proposition 2.5. *Let $\Omega \subset\subset \mathbb{C}P^n$ be a pseudoconvex domain with C^2 boundary $b\Omega$. Let $t_0 = t_0(\Omega)$ be the order of plurisubharmonicity for the distance function δ defined by (0.7). For any $0 \leq t < t_0$, $f \in L^2_{A'}(\delta^{-t})$, where $0 \leq p \leq n$ and $1 \leq q \leq n$, such that $\bar\partial f = 0$ in Ω, then $u = \bar\partial^* N f$ is in $L^2_{(p,q-1)}(\delta^{-t})$ with $\bar\partial u = f$ and*

$$\|u\|^2_{(-t)} \leq \tilde{C}_t |||f|||^2_{(-t)'},$$

where the weighted norms are given by (2.10.1)-(2.10.4).

Proof. Let $v = u\delta^{-t}$. We have that $\bar\partial^*_t v = 0$ if $q > 1$ and $v \perp \mathrm{Ker}(\bar\partial)$ in $L^2(\delta^t)$ for $q = 0$. This implies that $v = \bar\partial^*_t N_t \bar\partial v$. By (2.8) (with $\epsilon = 0$), we have that

$$|\frac{\bar\partial\delta}{\delta} \wedge u|^2_{(i\partial\bar\partial\phi)'} \leq \frac{1}{t_0}|\frac{\bar\partial\delta}{\delta} \wedge u|^2_{A'} \leq \frac{1}{t_0}|u|^2. \tag{2.18}$$

Thus for any $\epsilon > 0$, by Proposition 2.3 (the proof of (2.14)), we have

$$\begin{aligned}
\int_\Omega |u|^2\delta^{-t} &= \int_\Omega |v|^2\delta^t \leq \frac{1}{C_0 t}\int_\Omega |\bar\partial v|^2_{(i\partial\bar\partial\phi)'}\delta^t \\
&\leq (1+\frac{1}{\epsilon})\frac{1}{C_0 t}\int_\Omega |\bar\partial u|^2_{(i\partial\bar\partial\phi)'}\delta^{-t} + (1+\epsilon)\frac{t}{t_0}\int_\Omega |\frac{\bar\partial\delta}{\delta}\wedge u|^2_{A'}\delta^{-t} \\
&\leq (1+\frac{1}{\epsilon})\frac{1}{C_0 t}\int_\Omega |\bar\partial u|^2_{A'}\delta^{-t} + (1+\epsilon)\frac{t}{t_0}\int_\Omega |u|^2\delta^{-t}.
\end{aligned} \tag{2.19}$$

Choosing ϵ sufficiently small such that $(1+\epsilon)\frac{t}{t_0} = r < 1$ in (2.19), we conclude that

$$\int_\Omega |u|^2\delta^{-t} \leq \frac{(1+\frac{1}{\epsilon})\frac{1}{C_0 t}}{1-r}\int_\Omega |\bar\partial u|^2_{A'}\delta^{-t} \leq \tilde{C}_t|||f|||^2_{(-t)'}. \tag{2.20}$$

Proposition 2.5 is proved. □

We arrive at the main result of this section.

Theorem 2.6. *Let $\Omega \subset\subset \mathbb{C}P^n$ be a pseudoconvex domain with C^2-smooth boundary $b\Omega$. Then $\Box_{(p,q)}$ has closed range and the $\bar\partial$-Neumann operator $N_{(p,q)} : L^2_{(p,q)}(\Omega) \to L^2_{(p,q)}(\Omega)$ exists for every p, q such that $0 \le p \le n, 0 \le q \le n$. Moreover, for any $f \in L^2_{(p,q)}(\Omega)$, we have*

$$f = \bar\partial\bar\partial^* N_{(p,q)} f \oplus \bar\partial^*\bar\partial N_{(0,q)} f, \quad 1 \le q \le n-1.$$
$$f = \bar\partial^*\bar\partial N_{(p,0)} f \oplus Pf, \quad q = 0,$$

where P is the projection from $L^2_{(p,0)}(\Omega)$ onto $L^2_{(p,0)}(\Omega) \cap Ker(\bar\partial)$ and

$$N_{(p,0)} = \bar\partial^* N^2_{(p,1)} \bar\partial.$$

In addition, $\bar\partial N$, $\bar\partial^* N$ and N are bounded linear operators on $L^2_{(p,q)}(\Omega)$.

Proof. The L^2-existence theorem for the $\bar\partial$-Neumann operator N on Ω follows from the L^2-existence of the solution u for the $\bar\partial$-equation, which is proved in Proposition 2.4. The proof of Theorem 2.6 follows from Lemma 1.3 using Proposition 2.4 and (2.17) (see e.g., proofs of Theorem 4.4.1 and Theorem 4.4.3 in [CS] or [Hö3].) $\square$

We remark that when $p = 0$, Theorem 2.6 holds for pseudoconvex domains not necessarily with C^2 boundary from Theorem 2.1.

3. Sobolev estimates for the $\bar\partial$-Neumann operator on pseudoconvex domains in $\mathbb{C}P^n$

In this section we prove that the $\bar\partial$-Neumann operator N is a *bounded* linear operator on Sobolev spaces $W^t_{(p,q)}(\Omega)$ for small $t > 0$. In order to derive some *a priori* estimates for the $\bar\partial$-equation, we need a variant of the Bochner-Kodaira-Morrey-Kohn-Hörmander formula (cf. Proposition 1.4) as follows:

Proposition 3.1. *Let Ω be a domain in $\mathbb{C}P^n$ with C^2-smooth boundary $b\Omega$. Let $\delta(x) = d(x, b\Omega)$ be the distance function and $\lambda = -(\delta)^t$ for some $t > 0$. For any $f \in C^1_{(p,q)}(\overline{\Omega}) \cap Dom(\bar\partial^*)$, where $0 \le p \le n$ and $1 \le q \le n$, we have*

$$\begin{aligned}\int_\Omega (-\lambda)|\bar\partial f|^2 + \int_\Omega (-\lambda)|\vartheta f|^2 + 2\Re(\vartheta f, \overline{f \llcorner (\bar\partial\lambda)_\#}) \\ = \int_\Omega (-\lambda)\left(|\overline{\nabla} f|^2 + \langle \Theta f, \bar f\rangle + \langle (i\partial\bar\partial\lambda) f, \bar f\rangle\right)\end{aligned} \tag{3.1.1}$$

where Θ and $\overline{\nabla}$ are the same as in Proposition 1.4 and $\Re(h)$ is the real part of h.

Proof. The proof follows from the same calculation as in Hörmander [Hö2]. There is no boundary term since $t > 0$. Recall that $\vartheta = \bar\partial^*$. A simple calculation shows that

$$\vartheta_\phi f = \bar\partial^*_\phi f = \bar\partial^* f - f \llcorner (\bar\partial\phi)_\# \tag{3.1.2}$$

holds for any (p,q)-form u with $q \geq 1$. On the other hand, by expanding the term $\delta^t i\partial\bar\partial(-\log\delta^t)$ we have

$$\delta^t i\partial\bar\partial(-\log\delta^t) = ti\partial\bar\partial(-\delta^t) + t^2\frac{i\partial\delta\wedge\bar\partial\delta}{\delta^2}. \tag{3.1.3}$$

Substituting (3.1.2)-(3.1.3) into (1.33) of Proposition 1.4 with $\phi = -t\log\delta$, we obtain (3.1.1). □

Using lemma 2.2 and Proposition 3.1, we have the following estimates:

Proposition 3.2. *Let Ω be a domain in $\mathbb{C}P^n$ with C^2 boundary $b\Omega$. Let δ be the distance function from $b\Omega$. Let $0 < t < 1$ such that $\lambda = -(\delta)^t$ satisfies condition (2.5). For any $f \in L^2_{(p,q)}(\Omega)\cap Dom(\bar\partial)\cap Dom(\bar\partial^*)$, where $0 \leq p \leq n$ and $1 \leq q \leq n$, we have*

$$\begin{aligned}&\int_\Omega(-\lambda)|\bar\partial f|^2 + \int_\Omega(-\lambda)|\vartheta f|^2\\ &\geq \hat C_t\left(\int_\Omega(-\lambda)|\overline\nabla f|^2 + \int_\Omega(-\lambda)|f|^2 + \int_\Omega(-\lambda)\frac{|f\llcorner(\bar\partial\rho)_\#|^2}{|\rho|^2}\right),\end{aligned} \tag{3.2}$$

where $\hat C_t$ is a constant number independent of f.

Proof. Let $f \in C^1_{(p,q)}(\overline\Omega)\cap \mathrm{Dom}(\bar\partial^*)$. By (2.5), we have

$$\langle(i\partial\bar\partial\lambda)f,\bar f\rangle \geq C_1\left(\int_\Omega(-\lambda)|f|^2 + \int_\Omega(-\lambda)\frac{|f\llcorner(\bar\partial\rho)_\#|^2}{|\rho|^2}\right). \tag{3.3}$$

Also, we have for any $\epsilon > 0$,

$$|2\Re(-\vartheta f, \overline{f\llcorner(\bar\partial\lambda)_\#})| \leq \frac{t}{\epsilon}\int_\Omega(-\lambda)|\vartheta f|^2 + t\epsilon\int_\Omega(-\lambda)\frac{|f\llcorner(\bar\partial\rho)_\#|^2}{|\rho|^2}. \tag{3.4}$$

Choosing ϵ sufficiently small, (3.2) follows from (3.1.1), (3.3) and Proposition 1.5. This proves the proposition for C^1-smooth forms.

Since $C^1_{(p,q)}(\overline\Omega)\cap\mathrm{Dom}(\bar\partial^*)$ is dense in $\mathrm{Dom}(\bar\partial)\cap\mathrm{Dom}(\bar\partial^*)$ in the graph norm (cf. Lemma 4.3.2 in [CS]), we have that $C^1_{(p,q)}(\overline\Omega)\cap\mathrm{Dom}(\bar\partial^*)$ is also dense in the graph norm with weights defined by $(\int_\Omega(-\lambda)|\bar\partial f|^2+\int_\Omega(-\lambda)|\vartheta f|^2+\int_\Omega(-\lambda)|f|^2)^{\frac12}$ from the Dominated Convergence Theorem. The proposition is proved by approximation of C^1-smooth forms. □

Recall that $u = \bar\partial^* Nf$ is a solution of $\bar\partial u = f$ with the least L^2-norm. By the estimate above, we have the following:

Corollary 3.3. *Let $\Omega \subset\subset \mathbb{C}P^n$ and t be the same as in Proposition 3.2. Then*

$$\| \bar{\partial}^* N_{(p,q)} f \|_{(t)} \leq C \| f \|_{(t)}, \quad f \in Ker(\bar{\partial}),\ 2 \leq q \leq n. \tag{3.5}$$

$$\| \bar{\partial} N_{(p,q)} f \|_{(t)} \leq C \| f \|_{(t)}, \quad f \in Ker(\bar{\partial}^*),\ 0 \leq q \leq n-1. \tag{3.6}$$

where C depends only on t.

Proof. Notice that $N = \Box^{-1}$ on the range of $\Box$. It follows that $Nf \in \mathrm{Dom}(\Box)$. Hence, by (1.25.3), we see that $\bar{\partial}^* Nf$ is in $\mathrm{Dom}(\bar{\partial}) \cap \mathrm{Dom}(\bar{\partial}^*)$. If $q \geq 2$, for any $f \in L^2_{(p,q)}(\Omega)$, $\bar{\partial}^* Nf$ is a $(p, q-1)$-form. Thus for $f \in \mathrm{Ker}(\bar{\partial})$, one gets from (3.2) that

$$\begin{aligned}
\int_\Omega (-\lambda)|f|^2 &= \int_\Omega (-\lambda)|\bar{\partial}\bar{\partial}^* Nf|^2 + \int_\Omega (-\lambda)|\bar{\partial}^*\bar{\partial}^* Nf|^2 \\
&\geq \hat{C}_t \int_\Omega (-\lambda)|\bar{\partial}^* Nf|^2,
\end{aligned}$$

where we used the following facts: $f = (\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial})Nf$, $\bar{\partial}^*\bar{\partial}^* f = 0$ and $\bar{\partial} Nf = N\bar{\partial} f = 0$ with $f \in \ker(\bar{\partial})$. This proves inequality (3.5) by choosing $C = \frac{1}{\hat{C}_t}$.

Similarly, for any $0 \leq q \leq n-1$, $\bar{\partial} Nf$ is in $\mathrm{Dom}(\bar{\partial}) \cap \mathrm{Dom}(\bar{\partial}^*)$. Substituting $\bar{\partial} N_{(p,q)} f$ into (3.2), for $f \in \mathrm{Ker}(\bar{\partial}^*)$, we have

$$\begin{aligned}
\int_\Omega (-\lambda)|f|^2 &= \int_\Omega (-\lambda)|\bar{\partial}\bar{\partial} Nf|^2 + \int_\Omega (-\lambda)|\bar{\partial}^*\bar{\partial} Nf|^2 \\
&\geq \hat{C}_t \int_\Omega (-\lambda)|\bar{\partial} Nf|^2.
\end{aligned}$$

This proves the corollary by choosing $C = \frac{1}{\hat{C}_t}$. $\Box$

Let $W^s_{(p,q)}(\Omega)$ be the Sobolev space with $-\frac{1}{2} < s < \frac{1}{2}$ and let $\| \ \|_{s(\Omega)}$ denote its norm. For any u in $\mathrm{Dom}(\bar{\partial}) \cap \mathrm{Dom}(\bar{\partial}^*)$, we have $u \in W^1(\Omega, \mathrm{loc})$. This implies that u satisfies an elliptic system and $u \in W^s(\Omega)$ for $-\frac{1}{2} < s < \frac{1}{2}$ if and only if

$$\|u\|^2_{(-2s)} = \int_\Omega \delta^{-2s}|u|^2 < \infty. \tag{3.7}$$

For a proof of this, see Theorem C.4 in the Appendix in [CS].

Using (3.7) or (2.9) and the estimates above, we are ready to prove the Sobolev boundary regularity result for the (generalized) Bergman operator $P : L^2_{(p,q)}(\Omega) \to L^2_{(p,q)}(\Omega) \cap \ker(\bar{\partial})$. From Theorem 2.6, we have $P = I - \bar{\partial}^* N_{(p,q+1)}\bar{\partial}$.

Theorem 3.4. *Let $\Omega \subset\subset \mathbb{C}P^n$ be a pseudoconvex domain with C^2-smooth boundary. Let $0 < t < t_0$ and $t_0 = t_0(\Omega)$ be given by (0.7). Then the Bergman projection operator P is bounded from $W^{\frac{t}{2}}_{(p,q)}(\Omega)$ to $W^{\frac{t}{2}}_{(p,q)}(\Omega)$, where $0 \le p \le n$ and $0 \le q \le n-1$.*

Proof. By Proposition 2.5, we have that $\bar\partial^* N$ is bounded on $\mathrm{Ker}(\bar\partial)$ with estimates

$$\| \bar\partial^* N_{(p,q)} f \|_{(-t)} \le C|||f|||_{(-t)'}, \quad f \in \mathrm{Ker}(\bar\partial),\ 1 \le q \le n-1. \tag{3.8}$$

Let P_t denote the Bergman projection with respect to the weighted space $L^2(\delta^t)$. For any $f, g \in L^2_{(p,0)}(\Omega)$ with $\bar\partial g = 0$, we have

$$(Pf, g) = (f, g) = (\delta^{-t} f, g)_{(t)} = (P_t \delta^{-t} f, g)_{(t)} = (\delta^t P_t \delta^{-t} f, g).$$

This implies that

$$\begin{aligned} P &= P^2 = P\delta^t P_t \delta^{-t} \\ &= (I - \bar\partial^* N \bar\partial)\delta^t P_t \delta^{-t} \\ &= \delta^t P_t \delta^{-t} - \bar\partial^* N(\bar\partial(\delta^t) \wedge P_t \delta^{-t}) \end{aligned} \tag{3.9}$$

since $\bar\partial P_t = 0$. For any $f \in L^2_{(p,q)}(\Omega)$,

$$\|\delta^t P_t \delta^{-t} f\|^2_{(-t)} \le \|P_t \delta^{-t} f\|^2_{(t)} \le \|\delta^{-t} f\|^2_{(t)} = \|f\|^2_{(-t)}. \tag{3.10}$$

By (3.8), we have

$$\begin{aligned} \| \bar\partial^* N(\bar\partial(\delta^t) \wedge P_t \delta^{-t} f) \|^2_{(-t)} &\le C|||\bar\partial(\delta^t) \wedge P_t \delta^{-t} f|||^2_{(-t)'} \\ &\le C \| \delta^{\frac{t}{2}} P_t \delta^{-t} f \|^2 = C \| P_t \delta^{-t} f \|^2_{(t)} \\ &\le C \| \delta^{-t} f \|^2_{(t)} = C \| f \|^2_{(-t)} \ . \end{aligned} \tag{3.11}$$

From (3.9)-(3.11), we get

$$\| Pf \|_{(-t)} \le C \| f \|_{(-t)} \ . \tag{3.12}$$

Note that $W^{\frac{t}{2}} \subset L^2(\delta^{-t})$ by (3.7) or (2.9). From (3.10), we get

$$\| Pf \|_{(-t)} \le C \| f \|_{(-t)} \le C_1 \| f \|_{\frac{t}{2}} \ . \tag{3.13.1}$$

Using (3.13.1) and (3.7) or (2.9), one obtains that the Bergman projection satisfies

$$\| Pf \|_{\frac{t}{2}} \le C_2 \| f \|_{\frac{t}{2}} \ . \tag{3.13.2}$$

Theorem 3.4 is proved. $\square$

Let us now study the Sobolev regularities for other operators: N, $\bar\partial N$ and $\bar\partial^* N$.

Theorem 3.5. *Let $\Omega \subset\subset \mathbb{C}P^n$ be a pseudoconvex domain with C^2-smooth boundary $b\Omega$. Let $t_0 = t_0(\Omega)$ be the order of plurisubharmonicity for the distance function δ defined by (0.7). For any $0 < t < t_0$, the $\bar\partial$-Neumann operator N is bounded from $W^{\frac{t}{2}}_{(p,q)}(\Omega)$ to $W^{\frac{t}{2}}_{(p,q)}(\Omega)$, where $0 \le p \le n$ and $0 \le q \le n-1$. We also have the following estimates: for any $f \in W^s_{(p,q)}(\Omega)$,*

$$\| Nf \|_{\frac{t}{2}(\Omega)} \le 2C^2 \| f \|_{\frac{t}{2}(\Omega)}, \tag{3.14}$$

$$\| \bar\partial^* Nf \|_{\frac{t}{2}(\Omega)} \le C \| f \|_{\frac{t}{2}(\Omega)}, \tag{3.15}$$

$$\| \bar\partial Nf \|_{\frac{t}{2}(\Omega)} \le C \| f \|_{\frac{t}{2}(\Omega)}, \tag{3.16}$$

where C depends only on t.

Proof. We already proved that the Bergman projection $P = I - \bar\partial^* N \bar\partial$ are bounded on $W^{\frac{t}{2}}$. It is easy to verify that, if $\mathcal{F}^*$ is the adjoint map of $\mathcal{F}$ with respect to the L^2-norm, then

$$\begin{aligned} \|\mathcal{F}u \|_{W^{\frac{t}{2}}(\Omega)} &= \sup_{v\in L^2} \{ \frac{(\mathcal{F}u, \bar v)_{L^2}}{\|v \|_{W^{-\frac{t}{2}}}} \} \\ &= \sup_{v\in L^2} \{ \frac{(u, \mathcal{F}^* \bar v)_{L^2}}{\|v \|_{W^{-\frac{t}{2}}}} \} \le \| \mathcal{F}^* \|_{W^{-\frac{t}{2}}(\Omega)} \|u\|_{W^{\frac{t}{2}}(\Omega)}, \end{aligned} \tag{3.17}$$

(compare with Lemma 1.3).

From the proof of Theorem 3.4, we have that the canonical solution satisfies (3.8). Since the Bergman projection P is self-adjoint on $L^2(\Omega)$, by Theorem 3.4 and (3.17), we see that the Bergman projection P is bounded on $W^{\pm\frac{t}{2}}_{(p,q)}(\Omega)$.

Inequality (3.15) follows easily from Theorem 3.4 and the boundedness of the Bergman projection on $W^{\frac{t}{2}}$ since $\bar\partial^* Nf = \bar\partial^* NPf$.

Let $\tilde P = \bar\partial^* N \bar\partial$ be the projection operator into Ker$\bar\partial^*$. Then $P = I - \bar\partial^* N \bar\partial = I - \tilde P$. It follows that $\bar\partial Nf = \bar\partial N \tilde P f$. Since the Bergman projection is self-adjoint, both P and $\tilde P$ satisfy

$$\| Pf \|_{(t)} + \| \tilde P f \|_{(t)} \le C_3 \| f \|_{(t)} . \tag{3.18}$$

Thus from Corollary 3.3, we have that both $\bar\partial N$ and $\bar\partial^* N$ are bounded with estimates

$$\| \bar\partial Nf \|_{(t)} = \| \bar\partial N \tilde P f \|_{(t)} \le C_4 \| \tilde P f \|_{(t)} \le C_4 C_3 \| f \|_{(t)}, \ q \ge 0 \tag{3.19}$$

$$(3.20)\qquad \| \bar\partial^* Nf \|_{(t)} = \| \bar\partial^* NPf \|_{(t)} \leq C_4 \| Pf \|_{(t)} \leq C_4 C_3 \| f \|_{(t)},\ q \geq 2.$$

In fact (3.20) also holds for $q = 1$. This follows from the formula $\bar\partial^* Nf = \bar\partial_t^* N_t f - P_t \bar\partial_t^* N_t f$ for any $f \in \mathrm{Ker}(\bar\partial)$. Thus for any $f \in L^2_{(p,1)}(\Omega)$, by (2.14) and (2.10.4) we have

$$(3.21)\qquad \begin{aligned} \| \bar\partial^* Nf \|_{(t)} &= \| \bar\partial^* NPf \|_{(t)} \\ &= \| \bar\partial_t^* N_t Pf - P_t \bar\partial_t^* N_t Pf \|_{(t)} \leq C_5 \| Pf \|_{(t)} \leq C_5 C_3 \| f \|_{(t)} . \end{aligned}$$

Notice that $(\bar\partial N)^* = N\bar\partial^* = \bar\partial^* N$ and $(\bar\partial^* N)^* = N\bar\partial = \bar\partial N$. By (3.17)-(3.21), we obtain that both $\bar\partial N$ and $\bar\partial^* N$ are bounded operators on $W^{\pm\frac{t}{2}}_{(p,q)}(\Omega)$. Thus, (3.15)-(3.16) are true, by choosing $C = \max\{C_3C_4, C_3C_5\}$.

It remains to verify (3.14). By an observation of Range, we have

$$(3.22)\qquad N = \bar\partial\bar\partial^* N^2 + \bar\partial^*\bar\partial N^2 = \bar\partial N \bar\partial^* N + \bar\partial^* N \bar\partial N.$$

It follows from (3.15)-(3.16) and (3.22) that

$$(3.23)\qquad \| Nf \|_{(t)} \leq 2C^2 \| f \|_{(t)},\ q \geq 0.$$

By (3.23) and (3.17), we conclude that

$$(3.24)\qquad \| Nf \|_{(-t)} \leq 2C^2 \| f \|_{(-t)},\ q \geq 1,$$

since N is self-adjoint. Thus (3.14) now follows from (3.24) and (3.7) or (2.9). $\square$

Theorem 2 is a direct consequence of Theorems 3.4-3.5.

4. $\bar\partial$-closed extension from pseudoconvex boundaries in $\mathbb{C}P^n$

In this section we study the extension of $\bar\partial_b$-closed forms from the boundary of a pseudoconvex domain in $\mathbb{C}P^n$. This is equivalent to solving the $\bar\partial$-Cauchy problem on pseudoconvex domains, which is the dual of the $\bar\partial$-Neumann problem.

Recall that if $\bar\partial_b v^{(0,1)} = 0$ on $b\Omega$ and if $w^{(0,1)}$ is an arbitrary extension of $v^{(0,1)}$ on Ω, then we can correct $w^{(0,1)}$ to be a $\bar\partial$-closed extension $\tilde v^{(0,1)}$ on Ω by setting

$$(4.0)\qquad \tilde v^{(0,1)} = w^{(0,1)} + \star\bar\partial N_{(n,n-2)} \star(\bar\partial w^{(0,1)}) \text{ on } \Omega,$$

where $\star = \bar * : L^2_{(p,q)}(\overline{\Omega})$ is the Hodge star operator defined by

$$(4.1)\qquad \langle \phi, \bar\psi \rangle dV = \phi \wedge \star\bar\psi.$$

Notice that $\star = \bar *$ satisfies $\star(\lambda u) = \bar\lambda(\star u)$ for any complex number $\lambda \in \mathbb{C}$.

There are two issues in the application of the formula (4.0) above. The first one is that we require $\bar{\partial} w^{(0,1)} \in L^2$ when we apply Theorem 3.5 to the $(n, n-2)$-form $[\star(\bar{\partial} w)]$. Therefore, in some results stated below, we require that $w \in W^1(\Omega)$ or equivalently $v = w|_{b\Omega} \in W^{\frac{1}{2}}(b\Omega)$, (cf. Proposition 4.3).

The second difficulty occurs when $[\star(\bar{\partial} w)]$ is an $(n, 0)$-form, i.e., $n = 2$. It is easy to see that, for a (p, q)-form v on M with $q < n-1$, the $\bar{\partial}$-closed extension $\tilde{v}$ on $\mathbb{C}P^n$ exists *only if* $\bar{\partial}_b v = 0$. However, for all $(p, n-1)$-forms v on M, the equation $\bar{\partial}_b v = 0$ always holds. There is another necessary condition on a $(p, n-1)$-form v so that its $\bar{\partial}$-closed extension on Ω exists, see (4.9) (or equivalently (4.3)) below. In order to show that (4.9) holds for any $(p, n-1)$-form v on the Levi-flat hypersurface M, we derive a new Liouville type theorem for a pseudoconcave boundary $M = b\Omega$, see Proposition 4.5 below.

To find a $\bar{\partial}$-closed extension $\tilde{v}$ on $\mathbb{C}P^n$ for v with $\bar{\partial}_b v = 0$ on $b\Omega$, it suffices to solve the inhomogeneous equation $\bar{\partial} u = f$ with compact support $\mathrm{Supp}(u) \subset \overline{\Omega}$.

Proposition 4.1. *Let Ω be a pseudoconvex domain with C^2 boundary in $\mathbb{C}P^n$, $n \geq 2$. For every $f \in L^2_{(p,q)}(\mathbb{C}P^n)$, where $0 \leq p \leq n$ and $1 \leq q \leq n-1$, with $\bar{\partial} f = 0$ in the distribution sense in $\mathbb{C}P^n$ and f supported in $\overline{\Omega}$, one can find $u \in L^2_{(p,q-1)}(\mathbb{C}P^n)$ such that $\bar{\partial} u = f$ in the distribution sense in $\mathbb{C}P^n$ with u supported in $\overline{\Omega}$ and*

$$\int\limits_{\Omega} |\, u \,|^2 \, dV \leq C \int\limits_{\Omega} |\, f \,|^2 \, dV$$

for some $C > 0$.

Proof. From Theorem 2.6, the $\bar{\partial}$-Neumann operator of degree $(n-p, n-q)$ in Ω, denoted by $N_{(n-p,n-q)}$, exists. Let $\star : L^2_{(p,q)}(\Omega)$ be the Hodge star operator defined as in (4.1) above. We define

$$u = -\star\bar{\partial} N_{(n-p,n-q)} \star f = -\bar{*}\bar{\partial} N_{(n-p,n-q)} \overline{*f}, \tag{4.2}$$

then $u \in L^2_{(p,q-1)}(\Omega)$. By the proof of Corollary 3.3, we can verify that $\star u = \bar{\partial} N(\star f) \in \mathrm{Dom}(\bar{\partial}^*)$, because $N = \Box^{-1}$ on $\mathrm{Range}(\Box)$ and $\mathrm{Range}(N) \subset \mathrm{Dom}(\Box)$. Since $(\star u) \in \mathrm{Dom}(\bar{\partial}^*)$, it follows from (1.25.2) that $u|_{T^{(0,1)}(b\Omega)} = 0$. Extending u to $\mathbb{C}P^n$ by defining $u = 0$ in $\mathbb{C}P^n \setminus \Omega$, we obtain that u has support in $\overline{\Omega}$. It follows from a theorem of Kohn-Rossi [KoR] (see also Theorem 9.1.2 in Chen-Shaw [CS]) that $\bar{\partial} u = f$ in the distribution sense in $\mathbb{C}P^n$. $\Box$

In order to solve the $\bar{\partial}$-equation with compact support when $q = n$, there is another compatibility condition (see (4.3) below) and we have the following result:

Proposition 4.2. *Let Ω be a pseudoconvex domain with C^2-smooth boundary in $\mathbb{C}P^n$, $n \geq 2$. For any $f \in L^2_{(p,n)}(\mathbb{C}P^n)$, $0 \leq p \leq n$, such that f is supported in $\overline{\Omega}$ and*

$$\int_{\Omega} f \wedge g = 0 \qquad \textit{for every} \quad g \in L^2_{(n-p,0)}(\Omega) \cap Ker(\bar{\partial}), \tag{4.3}$$

one can find $u \in L^2_{(p,n-1)}(\mathbb{C}P^n)$ such that $\bar\partial u = f$ in the distribution sense in $\mathbb{C}P^n$ with u supported in $\overline{\Omega}$ and

$$\int_\Omega | u |^2 \, dV \le C \int_\Omega | f |^2 \, dV,$$

for some $C > 0$.

Proof. Using Theorem 2.6, the $\bar\partial$-Neumann operator $N_{(p,0)}$ exists for any $0 \le p \le n$ and we have

$$N_{(p,0)} = \bar\partial^* N^2_{(p,1)} \bar\partial. \tag{4.4}$$

The Bergman projection operator $P_{(p,0)}$ is given by

$$\bar\partial^* \bar\partial N_{(p,0)} = I - P_{(p,0)}. \tag{4.5}$$

We define u by

$$u = - \star \bar\partial N_{(n-p,0)} \star f. \tag{4.6}$$

$$\begin{aligned} \bar\partial u &= (-1)^{p+n} \star \bar\partial^* \bar\partial N_{(n-p,0)} \star f \\ &= f - (-1)^{p+n} \star P_{(n-p,0)} \star f. \end{aligned} \tag{4.7}$$

From (4.3), we get for any $g \in L^2_{(n-p,0)}(\Omega) \cap \mathrm{Ker}(\bar\partial)$,

$$(\star f, \bar g) = (-1)^{p+n} \overline{\int_\Omega g \wedge f} = 0.$$

Thus $P_{(n-p,0)}(\star f) = 0$ and $\bar\partial u = f$ in Ω. Using $\star u \in \mathrm{Dom}(\bar\partial^*)$ and extending u to be zero outside Ω, then $\bar\partial u = f$ in $\mathbb{C}P^n$ in the distribution sense. $\square$

Let us now summarize the necessary and sufficient condition on $f \in W^{\frac{1}{2}}_{(p,q)}(b\Omega)$ to have a $\bar\partial$-closed extension F on Ω.

Proposition 4.3. *Let $\Omega \subset\subset \mathbb{C}P^n$ be a pseudoconvex domain with C^2 boundary $M = b\Omega$. Let $f \in W^{\frac{1}{2}}_{(p,q)}(b\Omega)$, where $0 \le p \le n$, $1 \le q \le n-1$. We assume that*

$$\bar\partial_b f = 0, \qquad \textit{if } 1 \le q < n-1, \tag{4.8}$$

and

$$\int_M f \wedge \psi = 0 \quad \textit{for every } \psi \in L^2_{(n-p,0)}(\overline{\Omega}) \cap Ker(\bar\partial), \quad \textit{if } q = n-1. \tag{4.9}$$

Then there exists $F \in L^2_{(p,q-1)}(\Omega)$ such that $F = f$ on $b\Omega$ and $\bar\partial F = 0$ in Ω.

We remark that any $\psi \in L^2_{(n-p,0)}(\Omega) \cap \mathrm{Ker}(\bar\partial)$ has boundary values on $b\Omega$ in the $W^{-\frac{1}{2}}(b\Omega)$ space (see e.g. Lemma 2.1 in Michel-Shaw [MS2]). Thus the pairing (4.9) is well-defined in the sense of currents.

Proof. Since $f \in W^{\frac{1}{2}}_{(p,q)}(b\Omega)$ is a form, one can extend $f = \Sigma_{|I|=p,|J|=q} f_{I,\bar J}$ componentwise to Ω such that each component $f_{I,\bar J}$ is in $W^1(\Omega)$. For detailed construction of such an extension, see e.g. Lemma 9.3.3 in [CS]. Let $\tilde f$ be an arbitrary extension of f with $\tilde f \in W^1_{(p,q)}(\Omega)$

We first assume that $q+1 < n$. From (4.8), we can require that $f_1 = \bar\partial_b \tilde f = 0$ in M. If we extend f_1 to be zero outside Ω, we get $\bar\partial f_1 = 0$ in $\mathbb{C}P^n$ in the distribution sense. we set $v_0 = -\star\bar\partial N_{(n-p,n-q-1)}\star f_1$. From Proposition 4.1 and its proof, we have $v_0 \in L^2_{(p,q)}(\Omega)$. We set $v_0 = 0$ outside Ω. Then $\bar\partial v_0 = f_1$ in the sense of distributions on $\mathbb{C}P^n$ with $\mathrm{supp}(v_0) \subset \overline{\Omega}$.

Setting $F = \tilde f - v_0$ in Ω, we have $F = f$ on $b\Omega$ and $\bar\partial F = 0$ in Ω. This proves the proposition for $q < n-1$.

When $q = n-1$ and f satisfies (4.9), we let $f_1 = \bar\partial \tilde f$. Then for any $\psi \in C^1_{(n,0)}(\overline{\Omega})$ with $\bar\partial\psi = 0$, we have

$$\int_\Omega f_1 \wedge \psi = \int_\Omega \bar\partial \tilde f \wedge \psi = \int_{b\Omega} f \wedge \psi = 0. \tag{4.10}$$

When $\psi \in L^2_{(n,0)}(\Omega) \cap \ker(\bar\partial)$, (4.10) follows from approximating ψ by smooth forms. Let us now apply Proposition 4.2 to f_1, since f_1 satisfies (4.3) by (4.10). Setting $v_0 = -\star\bar\partial N_{(n-p,n-q-1)}\star f_1$ and $F = \tilde f - v_0$ as above, by the proof of Proposition 4.2, we can conclude the proof for the case of $q = n-1$ as well. $\square$

Recall that the domain Ω_+ is pseudoconcave if and only if its complement $\Omega_- = \mathbb{C}P^n \setminus \overline{\Omega}_+$ is pseudoconvex. In order to show that the condition (4.9) automatically holds on the Levi-flat boundary $M = b\Omega_\pm$, we observe that M is the boundary of a *pseudoconcave* domain as well. Indeed, if $\Omega_+ \cup \Omega_- = \mathbb{C}P^n \setminus M$ and if M is Levi-flat, then $\Omega_\pm$ is both pseudoconvex and pseudoconcave.

Notice that, by a theorem of Chern-Lashof, any compact domain $\bar\Omega$ with C^2-smooth boundary $b\Omega$ in $\mathbb{C}^n$ must have a point Q_0 in $b\Omega$ with strictly positive principle curvature. To see this fact, one considers the Gauss map from $b\Omega$ to the unit sphere $S^{2n-1} \subset \mathbb{C}^n$. Therefore, this theorem of Chern-Lashof implies that there is *no* compact pseudoconcave domain $\overline{\Omega}$ in $\mathbb{C}^n$. However, the complex projective space $\mathbb{C}P^n$ contains a lot of pseudoconcave domains.

The classical Liouville Theorem states that if $f : \mathbb{C}^n \to \mathbb{C}$ is a bounded holomorphic function on $\mathbb{C}^n$ then f must be constant. We will extend the classical Liouville Theorem of functions to the (p,q)-forms on pseudoconcave domains Ω^+ as follows.

First we study the one-sided $\bar\partial$-closed extension from a pseudoconcave domain Ω_+ to the whole space $\mathbb{C}P^n$ by using (4.8).

Proposition 4.4. *Let $\Omega_+ \subset \mathbb{C}P^n$ be a pseudoconcave domain with C^2-smooth boundary. Suppose that $f \in W^1_{(p,q)}(\overline{\Omega}_+)$ with $\bar\partial f = 0$, where $0 \le p \le n$ and $1 \le q < n-1$. Then there exists a $\bar\partial$-closed form $F \in L^2_{(p,q)}(\mathbb{C}P^n)$ such that F is a $\bar\partial$-closed extension of f.*

Proof. Let $M = b\Omega_+$ be the C^2-smooth pseudoconcave boundary in $\mathbb{C}P^n$. We consider the so-called Fermi map (cf. [Cha]) along M as follows:

$$\begin{aligned} h: \quad & M \times \mathbb{R} \to \mathbb{C}P^n \\ & (Q_0, t) \to Exp_{Q_0}(t\nabla\rho), \end{aligned} \tag{4.11}$$

where Exp is the exponential map of $\mathbb{C}P^n$ and $\rho(x) = -d(x, b\Omega_-)$ for $x \in \Omega_-$. It is well-known that, for sufficiently small $\epsilon > 0$, the Fermi-map $h|_{M\times[-\epsilon,\epsilon]}$ is an embedding map.

Let us now choose an extension $w \in L^2_{(p,q)}(\mathbb{C}P^n)$ of f as follows: for $0 \ge t \ge -\epsilon$, let

$$w(h(Q_0,t)) = f(h(Q_0,-t)). \tag{4.12}$$

Thus, w is well-defined on $U_\epsilon(M)$. We further extend w to $[\Omega_- \setminus U_{\frac{\epsilon}{2}}(M)]$ so that $w|_{\overline{\Omega}_+} = f$ and $w \in W^1_{(p,q)}(\mathbb{C}P^n)$.

Recall that Ω_- is pseudoconvex by the assumption. Applying Proposition 4.1 to the equation

$$\bar\partial u = \bar\partial w \quad \text{on } \Omega_-,$$

we get a solution u with support in $\overline{\Omega}_-$ by choosing

$$u = -\star\bar\partial N_{(n-p,n-q-1)}(\star\bar\partial w) \qquad \text{on } \Omega_-. \tag{4.13}$$

Inspired by (4.0), we let

$$F_- = w - u = w + \star\bar\partial N_{(n-p,n-q-1)}(\star\bar\partial w) \qquad \text{on } \Omega_-, \tag{4.14}$$

where u is given by (4.13). A direct computation shows that

$$\bar\partial u = \bar\partial w - (-1)^{p+q} \star P_{(n-p,n-q)} \star (\bar\partial w), \tag{4.15}$$

where $P_{(n-p,n-q)} : L^2_{(p,q)}(\Omega_-) \to \mathrm{Ker}(\bar\partial)$ is the Bergman projection.

When w is a (p,q)-form with $q < n-1$, $\bar\partial w \in \mathrm{Ker}(\bar\partial)$ and hence $P(\star(\bar\partial w)) = 0$. By (4.15) and $P(\star(\bar\partial w)) = 0$, we have that $\bar\partial u = \bar\partial w$ and $\bar\partial F_- = 0$.

By the proof of Proposition 4.1, we know that u has compact support in $\overline{\Omega}_-$. Define

$$F = \begin{cases} f, & x \in \bar\Omega_+, \\ F_-, & x \in \Omega_-. \end{cases} \tag{4.16}$$

Then $F \in L^2_{(p,q)}(\mathbb{C}P^n)$, $F = f$ on $\bar\Omega_+$ and $\bar\partial F = 0$ in $\mathbb{C}P^n$ in the distribution sense. This proves the Proposition. $\square$

In order to show that (4.9) holds, it is sufficient to derive a new Liouville type Theorem, i.e., to show that $L^2_{(n-p,0)}(\Omega) \cap \mathrm{Ker}(\bar\partial) = 0$ for a pseudoconcave domain and $n - p > 0$.

Proposition 4.5. *Let $\Omega_+ \subset\subset \mathbb{C}P^n$ be a pseudoconcave domain with C^2-smooth boundary $b\Omega_+$. Then $L^2_{(p,0)}(\Omega_+) \cap Ker(\bar\partial) = \{0\}$ for every $0 < p \leq n$; and $L^2_{(0,0)}(\Omega_+) \cap Ker(\bar\partial) = \mathbb{C}$.*

Proof. We will extend f to be a $\bar\partial$-closed form in $\mathbb{C}P^n$ in a similar way as in the proof of Proposition 4.4. If $f \in L^2_{(p,0)}(\Omega_+) \cap \mathrm{Ker}(\bar\partial)$, then there exists a $\bar\partial$-closed extension $F \in L^2$ on the whole $\mathbb{C}P^n$. To see this, we extend f to be $\tilde f$ in $\mathbb{C}P^n$ such that $f_1 \equiv \bar\partial \tilde f \in L^2_{(p,1)}(\mathbb{C}P^n)$ with support $f_1 \subset \overline{\Omega}_-$, where $\Omega_- = \mathbb{C}P^n \setminus \overline{\Omega}_+$.

Such an L^2-extension $\tilde f$ can be carried out, because $b\Omega$ is a C^2-smooth real hypersurface. Here is the detail for the construction of $\tilde f$ and f_1. Let $h : b\Omega \times [-\epsilon, \epsilon] \to \mathbb{C}P^n$ be the Fermi-map given by (4.11). For sufficiently small ϵ, h is an embedding map which gives rise to the so-called Fermi coordinate along $b\Omega$. We now define $\tilde f(h(Q,t)) = f(h(Q,-t))$ for $h(Q,t) \in \Omega_-$ and $|t| < \epsilon/2$. Outside the annuli neighborhood $h(b\Omega \times [-\epsilon, \epsilon])$, we require $\tilde f$ to be C^2-smooth. Therefore, the $(p,1)$-form $f_1 \equiv \bar\partial \tilde f \in L^2_{(p,1)}(\mathbb{C}P^n)$ with support $f_1 \subset \overline{\Omega}_-$.

Using Propositions 4.1-4.4, we see that there exists a solution $v_0 \in L^2_{(p,0)}(\Omega_-)$ such that $\bar\partial v_0 = f_1$ in $\mathbb{C}P^n$ if we extend v_0 to be zero on Ω_+. Thus, the extension form $F = \tilde f - v_0$ is $\bar\partial$-closed in the whole space $\mathbb{C}P^n$. We observe that $L^2_{(p,0)}(\mathbb{C}P^n) \cap \mathrm{Ker}(\bar\partial) = \{0\}$ for all $0 < p \leq n$; and $L^2_{(0,0)}(\mathbb{C}P^n) \cap \mathrm{Ker}(\bar\partial) = \mathbb{C}$. Hence, $f = 0$ when $p > 0$, and f must be a constant when it is a L^2-holomorphic function on Ω_+. This proves Proposition 4.5. $\square$

Some related results similar to Propositions 4.4-4.5 were obtained by Henkin-Iordan [HI] earlier via a different argument. Let us now apply our results above to a C^2-smooth Levi-flat hypersurface M if it exists.

Corollary 4.6. *Let $M \subset \mathbb{C}P^n$ be a C^2-smooth Levi-flat hypersurface. For any $f \in W^{\frac{1}{2}}_{(p,n-1)}(M)$, where $0 \leq p < n$, there exists a $\bar\partial$-closed form $F \in L^2_{(p,n-1)}(\mathbb{C}P^n)$ such that F is a $\bar\partial$-closed extension of f.*

Proof. Let M be a compact connected C^2-smooth real hypersurface in $\mathbb{C}P^n$. It is well-known that $f \in W^{\frac{1}{2}}_{(p,q)}(M)$ if and only if there exists an extension $\tilde f$ of f with $\tilde f \in W^1_{(p,q)}(\mathbb{C}P^n)$.

Since $\mathbb{C}P^n$ is simply-connected, $\mathbb{C}P^n \setminus M = \Omega_+ \cup \Omega_-$ has exactly two connected components: Ω_+ and Ω_-. Since $\Omega_\pm$ is pseudoconvex, we let

$$F_\pm = \tilde f + \star[\bar\partial N_{(n-p,0)}|_{\Omega_\pm}](\star \bar\partial \tilde f) \qquad \text{on } \Omega_\pm \tag{4.17}$$

and $v_\pm = -\star \bar\partial N_{(n-p,0)}(\star \bar\partial \tilde f)$. By Theorem 2.6, we have $F_\pm \in L^2_{(p,n-1)}(\mathbb{C}P^n)$.

Since M is Levi-flat, we see that both Ω_+ and Ω_- are pseudoconcave as well. By Proposition 4.5, the form $f_1 = \bar\partial \tilde f$ satisfies (4.3) and (4.9) on each of $\Omega_\pm$. By the proofs of Propositions 4.1-4.5, we conclude that F is the $\bar\partial$-closed extension of f on the whole $\mathbb{C}P^n$. $\square$

We are now ready to study the inhomogeneous equation (0.4).

Corollary 4.7. *Let $M \subset \mathbb{C}P^n$ be a C^2-smooth Levi-flat hypersurface. Let $f \in W^{\frac{1}{2}}_{(p,q)}(M)$, where $0 \le p \le n$, $1 \le q \le n-1$ and $p \neq q$. We assume that f is $\bar\partial_b$-closed in M if $q < n-1$ and $p < n-1$ if $q = n-1$. Then there exists $u \in W^{\frac{1}{2}}_{(p,q-1)}(M)$ such that $\bar\partial_b u = f$ in M.*

Proof. From Proposition 4.4 (for $q < n-1$) and Corollary 4.6 (for $q = n-1$), there exists a $\bar\partial$-closed $F \in L^2_{(p,q)}(\mathbb{C}P^n)$ such that $F = f$ on M.

From our assumption $p \neq q$, we have that $H^{p,q}(\mathbb{C}P^n) = \{0\}$. This implies that there exists an inverse $\mathbb{G}$, the Green operator, for $\square$ on $\mathbb{C}P^n$ such that $\square\mathbb{G} = I$ on $L^2_{(p,q)}(\mathbb{C}P^n)$ and the the Hodge decomposition theorem holds for (p,q)-forms on $\mathbb{C}P^n$. For any $\bar\partial$-closed (p,q)-form F, we have

$$F = \bar\partial\bar\partial^*\mathbb{G}F + \bar\partial^*\bar\partial\mathbb{G}F = \bar\partial\bar\partial^*\mathbb{G}F + \bar\partial^*\mathbb{G}\bar\partial F = \bar\partial\bar\partial^*\mathbb{G}F. \tag{4.18}$$

Thus $F = \bar\partial\tilde u$ where $\tilde u = \bar\partial^*\mathbb{G}F$ is the canonical solution on $\mathbb{C}P^n$. Using the interior regularity for $\bar\partial$, we see that $\tilde u \in W^1(\mathbb{C}P^n)$. Restricting $\tilde u$ to M and denoting the restriction by u, we have $u \in W^{\frac{1}{2}}(M)$ and $\bar\partial_b u = f$.

5. Proof of Theorem 1

We will use results of Section 4 to prove Theorem 1. Let M be a $C^{2,\alpha}$ Levi-flat hypersurface in $\mathbb{C}P^n$, $n \ge 2$ and let $\rho = -\delta$ be the signed distance function for M. We first discuss solvability with regularity for the $\bar\partial_b$ equation in the Sobolev spaces on M.

As we pointed out in Section 0, the proof of Theorem 1 is reduced to a problem of finding a continuous solution u of

$$i\partial_b\bar\partial_b u = f_b = f|_{[T(M)]_{\mathbb{R}}\cap J[T(M)]_{\mathbb{R}}} \quad \text{on } M^{2n-1} \tag{5.1}$$

on M under the assumption that $f = dv$ is an exact form and f_b is a real-valued Hölder continuous (1, 1)-form. When f_b is a (1, 1)-form and $v^{(0,1)}$ is part of v, we have $\bar\partial_b v^{(0,1)} = 0$ and $\partial_b v^{(1,0)} = 0$.

There is the corresponding classical Lelong equation

$$i\partial\bar\partial\tilde u = \tilde f \quad \text{on } \mathbb{C}P^n \tag{5.2}$$

where $\tilde f = d\tilde v$ is an exact real-valued $(1,1)$-form on $\mathbb{C}P^n$. Since $\tilde f$ is of type $(1,1)$, we have $\bar\partial\tilde v^{(0,1)} = 0$ and $\partial\tilde v^{(1,0)} = 0$ where $\tilde v = \tilde v^{(0,1)} + \tilde v^{(1,0)}$ on $\mathbb{C}P^n$. The $\partial\bar\partial$-exact Poincaré Lemma states that if $\mathbb{G}_{(0,1)}$ is the Green's operator of the Laplace operator $\Box_{(0,1)}$ acting on $(0,1)$-forms of $\mathbb{C}P^n$, then $\tilde{u} = 2\Re\{\bar{\partial}^*\mathbb{G}_{(0,1)}\tilde{v}^{(0,1)}\}$ on $\mathbb{C}P^n$ is a solution of (5.2), where $\Re\{\phi\}$ denotes the real part of ϕ (cf. [Zh]).

Hence, to solve our equation (5.1) it is sufficient to extend the $\bar{\partial}_b$-closed 1-form $v^{(0,1)}$ of M to be a global $\bar{\partial}$-closed form $\tilde{v}^{(0,1)}$ on the whole space $\mathbb{C}P^n$. Recall that if $w_{\pm}^{(0,1)}$ is an *arbitrary* extension of $v^{(0,1)}$ from $M = b\Omega_{\pm}$ to $\Omega_{\pm}$, the $\bar{\partial}$-closed extension $\tilde{v}_{\pm}^{(0,1)}$ of $v^{(0,1)}$ is given by

$$\tilde{v}_{\pm}^{(0,1)} = w_{\pm}^{(0,1)} + [\bar{*}(\bar{\partial}N_{(n,n-2)}|_{\Omega_{\pm}})]\bar{*}(\bar{\partial}w_{\pm}^{(0,1)}).$$

We would also like to explain why the $W^s(\Omega)$-regularity result of Theorem 2 is good enough for the proof of Theorem 1. When the Levi-flat hypersurface M is $C^{2,\alpha}$-smooth, we can choose $w = \theta$ to be the connection form for the complex line bundle of equidistant hypersurfaces in a neighborhood $U_\epsilon(M)$ of M. It was shown in Section 1 that both w and dw (i.e., θ and the curvature $\Theta^N = d\theta$) are $C^{0,\alpha}$-smooth in $U_\epsilon(M)$. If Theorem 2 holds, the $\bar{\partial}$-closed extension $\tilde{v}^{(0,1)}$ of $\theta^{(0,1)}$ is in $W^s_{(0,1)}(\mathbb{C}P^n)$. Using elliptic theory on $\mathbb{C}P^n$, we obtain that the solution $\tilde{h} = \tilde{u}$ to (5.2) is in $W^{1+s}_{(0,0)}(\mathbb{C}P^n)$, see [GT]. Applying the trace theorem in Sobolev spaces to $\tilde{h}$, we conclude that $h = \tilde{h}|_M$ is in $W^{\frac{1}{2}+s}(M)$. By the classical elliptic theory on the holomorphic leaves of M, we already know that h is C^∞ in each holomorphic leaf of M. With some extra efforts, we can show that h is Hölder continuous in remaining directions of $[T(M)]_{\mathbb{R}}$, whenever $h \in W^{\frac{1}{2}+s}(M)$ and $i\partial_b\bar{\partial}_b h = \Theta_b \in C^{0,\alpha}(M)$ for some $s, \alpha > 0$, see Lemmas 5.1-5.2 below.

Lemma 5.1. *Let $M \subset \mathbb{C}P^n$ be a C^2-smooth Levi-flat hypersurface and let $f \in W^{\frac{1}{2}}_{(p,q)}(M)$, where $0 \le p \le n$, $1 \le q \le n-1$ and $p \ne q$. Suppose that $\epsilon_0 = \frac{1}{2}\min\{t_0(\Omega_+), t_0(\Omega_-)\}$ and $t_0(\Omega)$ is the order of plurisubharmonicity of Ω given by Definition 0.1. We further assume that $f \in W^{\frac{1}{2}+\epsilon}_{(p,q)}(M)$ is $\bar{\partial}_b$-closed in M if $q < n-1$ and $p < n-1$ if $q = n-1$. Then for any $0 \le \epsilon < \epsilon_0$, there exists $u \in W^{\frac{1}{2}+\epsilon}_{(p,q-1)}(M)$ satisfying $\bar{\partial}_b u = f$ in M.*

Proof. Let $\mathbb{C}P^n \setminus M = \Omega_+ \cup \Omega_-$. Let t_0^+ and t_0^- be the orders of plurisubharmonicity for the distance functions associated with Ω_+ and Ω_- respectively defined in Definition 0.1.

We use the same notion as in Section 4. We can construct a $\bar{\partial}$-closed form $F \in W^\epsilon_{(p,q)}(\mathbb{C}P^n)$ such that $F|_M = f$. This is proved by Theorem 3.5 and (4.14)-(4.18) as follows.

Since $f \in W^{\frac{1}{2}+\epsilon}_{(p,q)}(M)$, there exists an extension $\tilde{f} \in W^{1+\epsilon}(\mathbb{C}P^n)$ of f by the Trace Theorem (cf. Appendix of [CS]). As in the proof of Corollary 4.6, we let

$$F_{\pm} = \tilde{f} + \star[\bar{\partial}N_{(n-p,n-q-1)}|_{\Omega_{\pm}}](\star\bar{\partial}\tilde{f}) \qquad \text{on } \Omega_{\pm}.$$

This implies that $F \in W^{\epsilon}(\mathbb{C}P^n)$ by Theorem 3.5.

Next, we solve the equation $\bar\partial \tilde u = F$ on the whole manifold $\mathbb{C}P^n$. For the same reason as in the proof of Corollary 4.7, we conclude that $\tilde u \in W^{1+\epsilon}$ with $\bar\partial \tilde u = F$. This gives that $u = \tilde u|_M$ is in $W^{\frac{1}{2}+\epsilon}_{(p,q-1)}(M)$ by the Trace Theorem again. $\square$

We need two more preliminary results for the proof of Theorem 1. We denote by $\mathcal{N}^{1,0}_{\widetilde{\nabla\rho}}$ the complex line bundle and its associated curvature form by $\tilde\Theta^N$ as in Section 1. From Propositions 1.1-1.2, Θ_b is a $(1,1)$-form on the Levi-flat hypersurface M. As in Section 1, we let β be the real 1-form such that

$$\tilde\Theta^N = -d\theta = \sqrt{-1}d\beta \quad \text{ on } U_\epsilon(M),$$

where

$$\beta(\cdot) = \operatorname{Hess}(\rho)(\cdot, J(\nabla\rho)). \tag{5.3}$$

Write

$$\beta = \beta^{1,0} + \beta^{0,1}$$

where $\beta^{1,0}$ and $\beta^{0,1}$ are the (1,0) and (0,1) components of β.

Lemma 5.2. *Let M be a compact $C^{2,\alpha}$ Levi-flat hypersurface in $\mathbb{C}P^n$ and $\beta^{0,1}$ be the $(0,1)$-form defined in (5.3). Then there exists a function $u \in C^{\epsilon}(M)$ such that*

$$\bar\partial_b u = \beta^{0,1} \quad \text{in } M \tag{5.4}$$

for sufficiently small $\epsilon < \frac{1}{4}\min\{\alpha, t_0(\Omega_\pm)\}$, where $t_0(\Omega_\pm)$ is the plurisubharmonicity of $\Omega_\pm$ and $\Omega_+ \cup \Omega_- = \mathbb{C}P^n$.

Proof. Since ρ is of class $C^{2,\alpha}$, the 1-form β is obtained from the Hessian of ρ, hence, is in C^α. The Levi-flat hypersurface M is locally foliated by complex manifolds $\{\Sigma_t\}$, where $V = \cup_{|t|<\mu}\Sigma_t$ is an open subset of M^{2n-1}. Since each leaf Σ_t is a complex hypersurface, it follows from the Chern formula that the curvature tensor $\Theta_b = \tilde\Theta^N|_{\Sigma_t}$ is a $(1,1)$-form (see Proposition 1.2). Thus from (1.11) and type consideration, we get for $n > 2$,

$$\bar\partial_b \beta^{0,1} = 0 \quad \text{in } M. \tag{5.5}$$

When $n = 2$, $\beta^{0,1}$ satisfies the compatibility condition (4.9) by Proposition 4.5.

We claim that there exists a solution u of (5.4) such that $u \in W^{\frac{1}{2}+\epsilon}(M)$ for some $\epsilon > 0$. If $\alpha \ge \frac{1}{2}$, we can use Lemma 5.1 directly. If $\alpha < \frac{1}{2}$, we note that the Cartan-Chern-Gauss structure equation holds in a tubular neighborhood $U = U_\epsilon(M)$ of M. Thus, $\beta^{0,1} \in C^\alpha(U) \subset W^\alpha(U)$. Using the last assertion of Proposition 1.1, $\bar\partial\beta^{0,1}$ is in $C^\alpha(U)$ since it is the $(0,2)$-component of the curvature form $\tilde\Theta^N$, which is in

C^α. Thus $\beta^{0,1}$ has an extension to $\mathbb{C}P^n$ with $\bar\partial\beta^{0,1}$ in $W^\alpha_{(0,2)}(\Omega_+)$ and $W^\alpha_{(0,2)}(\Omega_-)$ respectively.

Repeating the same arguments as in the proof of Proposition 4.4 for $q < n-1$ and Corollary 4.6 for $q = n-1$, we obtain a $\bar\partial$-closed extension $F \in L^2$ of $\beta^{0,1}$ on the whole space $\mathbb{C}P^n$ with $F = \beta^{0,1}$ on M. Also from the boundary regularity of the $\bar\partial$-Neumann operator proved in Theorem 3.5, the extension F is in $W^\epsilon(\mathbb{C}P^n)$ with $\epsilon < \min\{\alpha, \epsilon_0\}$, where ϵ_0 is defined in Lemma 5.1.

Using the same arguments as in the proof of Corollary 4.7, we can prove that there exists $\tilde u \in W^{1+\epsilon}(\mathbb{C}P^n)$ satisfying $\bar\partial\tilde u = F$. Therefore, if $u = \tilde u|_M$, then we have $u \in W^{\frac12+\epsilon}(M)$ satisfying (5.4).

It remains to show that u is Hölder continuous in M. This follows from the following version of the Sobolev embedding theorem. We note that, on each leaf Σ_t, u satisfies an elliptic equation. Thus we already have that u is smooth on each leaf Σ_t, because $(\partial\bar\partial u)|_{\Sigma_t} = \Theta_b$ and Θ_b is analytic on each holomorphic leaf Σ_t. It remains only to show that u is Hölder continuous in the transversal direction $\frac{\partial}{\partial t}$. To do this we need to parametrize our hypersurface M locally.

Let $(z', g(z',t))$ denote the leaf L_t where $g(z',t)$ is holomorphic in $z' \in \mathbb{B}^{n-1} \subset \mathbb{C}^{n-1}$ and C^2-smooth in t. We can parametrize M locally as a graph of a function $\eta + g$, by setting

$$\Psi(z',t) = (z', \eta(t) + g(z',t)),$$

where $z' \in \mathbb{C}^n$, $0 \le |t| < \mu$ and $\eta(t)$ is a $C^{1,\alpha}$ function in t with $\eta(0) = 0$ and $\eta'(0) = 1$. Clearly, $\Psi : \mathbb{B}^{n-1} \times (-\mu,\mu) \to M$ is a local coordinate map of M, where $\mathbb{B}^{n-1} \subset \mathbb{C}^{n-1}$ is an open set of $\mathbb{C}^{n-1}$. Using a result of Barrett-Fornaess (see [BaF]), the foliation is actually $C^{2,\alpha}$.

It is easy to see that the push forwards $\bar L_i = \Psi_*(\frac{\partial}{\partial \bar z_i})$ of $\frac{\partial}{\partial\bar z_i}$, $i = 1, \cdots, n-1$, are the tangential Cauchy-Riemann equations for M. Let

$$T = \Psi_*(\frac{\partial}{\partial t}). \tag{5.6}$$

Since

$$\left[\frac{\partial}{\partial t}, \frac{\partial}{\partial \bar z_i}\right] = 0, \quad \left[\frac{\partial}{\partial t}, \frac{\partial}{\partial z_i}\right] = 0, \tag{5.7}$$

we have

$$[T, \bar L_i] = 0, \ [T, L_i] = 0.$$

Thus the tangential Cauchy-Riemann equations $\bar\partial_b$ are just the Cauchy-Riemann equations on Σ_t and they commute with T. When u is restricted to each leaf Σ_t, u satisfies an elliptic system in coordinates z' and

$$\partial_{z'}\bar\partial_{z'} u = \Theta_b \qquad \text{on } \Sigma_t, \tag{5.8}$$

where $\Theta_b = \tilde{\Theta}^N|_{[T(M)]_{\mathbb{R}} \cap J[T(M)]_{\mathbb{R}}} = \tilde{\Theta}^N|_{\Sigma_t}$ is a $(1,1)$-form and J is the complex structure of $\mathbb{C}P^n$, see Proposition 1.2.

From the classic Schauder theorem (cf. [GT]) for elliptic systems on $\mathbb{C}^{n-1}$, we get that $u \in C^{2,\alpha}(\Sigma_t)$ for each t since $\tilde{\Theta}^N$ is in C^α. Furthermore, we have that there exists a constant C_1 independent of t such that

$$|u|_{C^{2,\alpha}(\Sigma_t)} \leq C_1(|\tilde{\Theta}^N|_{C^\alpha(\Sigma_t)} + \|u\|_{L^2(\Sigma_t)}), \tag{5.9}$$

where C_1 depends on V and Ψ, but C_1 is independent of t, (because the local foliation is C^2-smooth and the equation (5.8) is uniformly elliptic on $\Sigma_t \subset V$ independent of t).

Recall that $V = \cup_{|t|<\mu} \Sigma_t \subset M$. From the Sobolev trace theorem, a function $u \in W^{\frac{1}{2}+\epsilon}(M)$ has L^2-trace on each leaf. Therefore, there exists $C_2 > 0$ independent of t such that

$$\|u\|_{L^2(\Sigma_t)} \leq C_2 \|u\|_{W^{\frac{1}{2}+\epsilon}(M)}. \tag{5.10}$$

Combining (5.9) and (5.10), we get

$$|u|_{L^\infty(V)} \leq \sup_{|t|<\mu} |u|_{C^{2,\alpha}(\Sigma_t)} \leq C_3. \tag{5.11}$$

Thus we have already proved that u is bounded.

It remains to show that u is Hölder continuous in t as well. To do this we differentiate the equation (5.8) in t with order $0 < \epsilon < 1$.

Let $D^\epsilon_{t,h}$ denote the finite difference

$$D^\epsilon_{t,h} u = \frac{u(z', t+h) - u(z', t)}{|h|^\epsilon}$$

and let $|\ |_s$ be the Besov norm given by

$$|u|_s = \sup_{0<|h|<\eta_0} \frac{\|u(x+h) - u(x)\|}{|h|^s}.$$

The Besov norm is weaker than the Sobolev norm $\|u\|_s$. It is easier to use the Besov norm than the Sobolev norm (see e.g. Hörmander [Hö4]) and we have

$$\|u\|_{s'} \leq |u|_s \leq \|u\|_s \tag{5.12}$$

for any $s' < s$. Let $0 < \epsilon' < \epsilon$. We claim that

$$\sup_{0<|h|<\eta_0} |D^{\epsilon'}_{t,h} u|_{\frac{1}{2}+\epsilon-\epsilon'} \leq C|u|_{\frac{1}{2}+\epsilon}. \tag{5.13}$$

Assuming (5.13) for the moment, we get from (5.12) that $D^{\epsilon'}_{t,h}u \in W^{s'}$ for any $\frac{1}{2} < s' < \frac{1}{2} + \epsilon - \epsilon'$. Note that from (5.7), the operators $D^{\epsilon'}_{t,h}$ and $\partial_b \bar{\partial}_b$ commute. Thus we have

$$\partial_b \bar{\partial}_b D^{\epsilon'}_{t,h} u = D^{\epsilon'}_{t,h} \partial_b \bar{\partial}_b u = D^{\epsilon'}_{t,h} \tilde{\Theta}^N \in C^{\alpha-\epsilon'}(U). \tag{5.14}$$

Applying the classical Schauder estimates to the equation (5.14) and repeating the above arguments used to obtain (5.11), we have

$$D^{\epsilon'}_{t,h} u \in L^\infty(U). \tag{5.15}$$

This implies that $u \in C^{\epsilon'}(U)$. To finish the proof of the lemma, it remains to prove the claim (5.13). $\square$

Lemma 5.3. *Let u be a function with $|u|_{s+\epsilon'} < \infty$ as above. Then*

$$\sup_{0<|h|<\eta_0} |D^{\epsilon'}_{t,h} u|_s \le C|u|_{s+\epsilon'}.$$

Proof. We identify $h = (0, \cdots, 0, h)$. We have

$$\begin{aligned} &\frac{\|D^{\epsilon'}_{t,h} u(x+\tilde{h}) - D^{\epsilon'}_{t,h} u(x)\|}{|\tilde{h}|^s} \\ &= \frac{\|(u(x+\tilde{h}+h) - u(x+\tilde{h})) - (u(x+h) - u(x))\|}{|h|^{\epsilon'}|\tilde{h}|^s} \\ &\le \frac{(|u(x+\tilde{h})|_{s+\epsilon'} + |u|_{s+\epsilon'})|h|^{s+\epsilon'}}{|h|^{\epsilon'}|\tilde{h}|^s} \\ &\le \frac{2|u|_{s+\epsilon'}|h|^{s+\epsilon'}}{|h|^{\epsilon'}|\tilde{h}|^s}. \end{aligned} \tag{5.16}$$

Similarly, we have

$$\begin{aligned} &\frac{\|D^{\epsilon'}_{t,h} u(x+\tilde{h}) - D^{\epsilon'}_{t,h} u(x)\|}{|\tilde{h}|^s} \\ &= \frac{\|(u(x+\tilde{h}+h) - u(x+h)) - (u(x+\tilde{h}) - u(x))\|}{|h|^{\epsilon'}|\tilde{h}|^s} \\ &\le \frac{(|u(x+h)|_{s+\epsilon'} + |u|_{s+\epsilon'})|\tilde{h}|^{s+\epsilon'}}{|h|^{\epsilon'}|\tilde{h}|^s} \\ &\le \frac{2|u|_{s+\epsilon'}|\tilde{h}|^{s+\epsilon'}}{|h|^{\epsilon'}|\tilde{h}|^s}. \end{aligned} \tag{5.17}$$

By (5.16) and (5.17), we conclude that

$$
\begin{aligned}
\sup_{0<|h|<\eta_0} |D^{\epsilon'}_{t,h}u|_s &= \sup_{0<|h|<\eta_0} \sup_{0<|\tilde h|<\eta_0} \frac{\|D^{\epsilon'}_{t,h}u(x+\tilde h) - D^{\epsilon'}_{t,h}u(x)\|}{|\tilde h|^s} \\
&\le C \sup_{0<|h|+|\tilde h|<2\eta_0} \min(|h|^{s+\epsilon'}, |\tilde h|^{s+\epsilon'}) \frac{|u|_{s+\epsilon'}}{|h|^{s'}|\tilde h|^s} \\
&\le C|u|_{s+\epsilon'}.
\end{aligned}
$$

This proves the lemma. □

Proof of Theorem 1. Let M be a Levi-flat hypersurface of class $C^{2,\alpha}$ in $\mathbb{C}P^n$, $n \ge 2$. Let u be obtained in Lemma 5.2 and $h = 2\mathrm{Im}u$, where $\mathrm{Im}u$ is the imaginary part of u. From Lemma 5.2, the function h is Hölder continuous on M. Since h is real-valued and M is compact, there exists a point Q_0 such that h assumes a global maximum at Q_0. On the other hand, we have

$$
\begin{aligned}
\Theta_b &= \tilde\Theta^N|_{[T(M)]_{\mathbb{R}}\cap J[T(M)]_{\mathbb{R}}} \\
&= (-d\beta)|_{[T(M)]_{\mathbb{R}}\cap J[T(M)]_{\mathbb{R}}} = -\partial_b\beta^{0,1} - \bar\partial_b\beta^{1,0} \\
&= -\partial_b\bar\partial_b u - \bar\partial_b\partial_b\bar u \\
&= \partial_b\bar\partial_b h
\end{aligned}
\tag{5.18}
$$

on M. From Proposition 1.2, the curvature form $i\Theta_b$ is a positive $(1,1)$-form on M. Although h is only $C^{0,\epsilon}$- continuous on M, its Hessian and $i\partial_b\bar\partial_b h$ can be computed by using the barrier functions, (e.g., see [Ca]). From (5.18) and Proposition 1.2, h is a strictly plurisubharmonic function when restricted to each leaf Σ_t of M. Hence, at the global maximum point $Q_0 \in M$ of h, we obtain a strictly plurisubharmonic function h on that particular leaf Σ_0 containing Q_0 as an interior maximum. This contradicts the maximum principle (cf. [Ca]). Thus there does not exist any Levi-flat hypersurface of class $C^{2,\alpha}$ in $\mathbb{C}P^n$. Theorem 1 is proved. □

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