DETERMINING A FUNCTION FROM ITS MEAN VALUES OVER A FAMILY OF SPHERES

DAVID FINCH, SARAH PATCH, AND RAKESH

Abstract. Suppose D is a bounded, connected, open set in R^n and f a smooth function on R^n with support in $\overline{D}$. We study the recovery of f from the mean values of f over spheres centered on a part or the whole boundary of D. For strictly convex $\overline{D}$ we prove uniqueness when the centers are restricted to an open subset of the boundary. We provide an inversion algorithm (with proof) when the the mean values are known for all spheres centered on the boundary of D, with radii in the interval $[0, diam(D)/2]$. We also give an inversion formula when D is a ball in R^n, $n \geq 3$ and odd, and the mean values are known for all spheres centered on the boundary.

1. Introduction

Wave propagation and integral geometry are the physical and mathematical underpinnings of many medical imaging modalities. To date, standard modalities measure the same type of output energy as was input to the system. Ultrasound systems send and receive ultrasound waves; CT systems send and receive X-ray radiation. Recent work on a hybrid imaging technique, thermoacoustic tomography (TCT), uses radiofrequency (RF) energy input at time t_0 and measures emitted ultrasound waves [18]-[20].

RF energy is deposited impulsively in time and uniformly throughout the imaging object, causing a small amount of thermal expansion. The premise is that cancerous masses absorb more RF energy than healthy tissue [17]. Cancerous masses preferentially absorb RF energy heat and expand more quickly than neighboring tissue, creating a pressure wave which is detected by ultrasound transducers at the edge of the object. Assuming constant sound speed, c, the sound waves detected at any point in time $t > t_0$ were generated by inclusions lying on the sphere of radius $c(t - t_0)$ centered at the transducer. Therefore, this imaging technique requires inversion of a generalized Radon transform, because integrals of the tissue's RF absorption coefficient are measured over surfaces of spheres.

A large part of this work was done during the Fall of 2001, at the MSRI, Berkeley, CA, where the authors were participating in a semester-long program on Inverse Problems. The authors would like to thank the organizers of this program, particularly David Eisenbud, Gunther Uhlmann, and the NSF, for organizing this program and providing financial support. David Finch and Rakesh were on sabbatical and would like to thank their respective universities for providing this opportunity and financial support. Rakesh also worked on this article when visiting Vanderbilt University during the Spring of 2002 and would like to thank the Mathematics Department at Vanderbilt University for its hospitality and financial support.

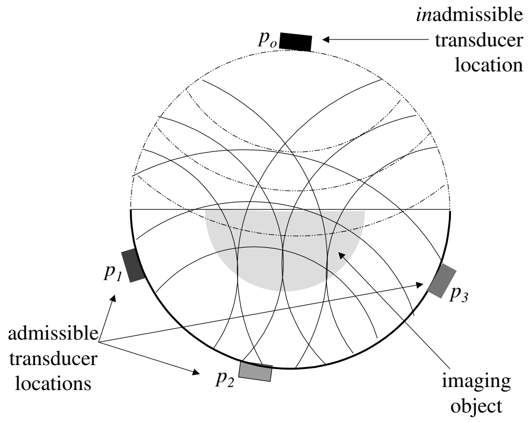


FIGURE 1. TCT mammography system

Figure 1 shows a TCT mammography system. The breast is immersed in a tank of water and transducers surround the exterior of the tank. Integrals of the RF absorption coefficient over spheres centered at each transducer are measured. Notice that only "limited angle" data may be measured, as we cannot put transducers on certain parts of the exterior of the tank.

The above motivated the study of the following mathematical problem. For a continuous, real valued function f on R^n, $n \geq 2$, p a point in R^n, and r a real number, define the mean value operator

$$(Mf)(p,r) = \frac{1}{w_n} \int_{|\theta|=1} f(p + r\theta)\, d\theta$$

where w_n is the surface area of the unit sphere in R^n. Let D denote a bounded, open, connected subset of R^n with a smooth boundary S. For functions f supported in D we are interested in recovering f from the mean value of f over spheres centered on S, that is given $(Mf)(p,r)$ for all p in S and all real numbers r we wish to recover f. We also examine the situation where the mean values of f are given over spheres centered on an open subset of S.

In the rest of the article, $B_\rho(p)$ will represent the open ball of radius ρ centered at p, $\overline{B_\rho(p)}$ its closure, and $S_\rho(p)$ its boundary; Ω^c will represent the complement of Ω. Further, all functions will be real valued.

We have the following results.

Theorem 1 (Uniqueness). *Suppose D is a bounded, open, subset of R^n, $n \geq 2$, with a smooth boundary S and $\overline{D}$ is strictly convex. Let Γ be any relatively open subset of S. If f is a smooth function on R^n, supported in $\overline{D}$, and $(Mf)(p,r) = 0$ for all $p \in \Gamma$ and all r, then $f = 0$.*

Here by the strict convexity of $\overline{D}$ we mean that if p, q are in $\overline{D}$ then any other point on the line segment pq is in D. Also, note that $(Mf)(p,r) = 0$ for all $p \in S$, $|r| > diam(D)$.

Theorem 2 (Reconstruction). *Suppose D is a bounded, open, connected subset of R^n, n odd and $n \geq 3$, with a smooth boundary S. If f is a smooth function on R^n, supported in $\overline{D}$, and $(Mf)(p,r)$ is known for all p in S and for all $r \in [0, diam(D)/2]$, then we may stably recover f. If $(Mf)(p,r)$ is known for all p in S and for all r, then f may be recovered by a simpler algorithm.*

Note that we do not assume $\overline{D}$ is convex but we do need the centers to vary over all of S. If D is a ball in R^n, $n \geq 3$ and n odd, and we know the mean values for all spheres centered on the boundary of D then we have an explicit inversion formula.

We introduce some notation to state the explicit inversion formula. Let $\tilde{C}^\infty(S_\rho(0) \times [0,\infty))$ consist of smooth functions $G(p,t)$ which are zero for t large and also $\partial_t^k G(p,t) = 0$ at $t = 0$ for $k = 0, 1, 2, \cdots$ and all $p \in S_\rho(0)$. Let us define the operator

$$\begin{aligned} \mathcal{N} \,:\, C_0^\infty(\overline{B_\rho(0)}) &\rightarrow \tilde{C}^\infty\left(S_\rho(0) \times [0,\infty)\right) \\ (\mathcal{N}f)(p,t) &= t^{n-2}(Mf)(p,t), \qquad p \in S_\rho(0),\ t \geq 0 \end{aligned}$$

and the operator (for odd $n \geq 3$)

$$\begin{aligned} \mathcal{D} \,:\, \tilde{C}^\infty(S_\rho(0) \times [0,\infty)) &\rightarrow \tilde{C}^\infty(S_\rho(0) \times [0,\infty)) \\ (\mathcal{D}G)(p,t) &= \left(\frac{1}{2t}\frac{\partial}{\partial t}\right)^{(n-3)/2} (G(p,t)). \end{aligned}$$

For example, $\mathcal{D} = I$ when $n = 3$.

We now compute the formal L^2 adjoints of $\mathcal{N}$ and $\mathcal{D}$. For $G \in \tilde{C}^\infty(S_\rho(0) \times [0,\infty))$, using the change of variables $(t,\theta) \rightarrow y = p + t\theta$, we note that

$$\begin{aligned} \langle \mathcal{N}f \,, G \rangle &= \int_{|p|=\rho} \int_0^\infty (\mathcal{N}f)(p,t)\, G(p,t)\, dt\, dS_p \\ &= \frac{1}{\omega_n} \int_{|p|=\rho} \int_0^\infty \int_{|\theta|=1} t^{n-2} f(p+t\theta)\, G(p,t)\, d\theta\, dt\, dS_p \\ &= \frac{1}{\omega_n} \int_{R^n} \int_{|p|=\rho} f(y)\, \frac{G(p,|p-y|)}{|p-y|}\, dS_p\, dy \\ &= \langle f \,, \mathcal{N}^* G \rangle, \end{aligned}$$

if we take

$$(\mathcal{N}^*G)(x) = \frac{1}{\omega_n} \int_{|p|=\rho} \frac{G(p,|p-x|)}{|p-x|}\, dS_p \,. \tag{1}$$

Note that for $G \in \tilde{C}^\infty(S_\rho(0) \times [0,\infty))$, $(\mathcal{N}^*G)(x)$ is a smooth function on R^n with compact support. The smoothness may be seen as follows; from the hypothesis on G, we may express $G(p,t)$ in the form $G(p,t) = tK(p,t^2)$ for $|p| = \rho$, $t \in [0,\infty)$ for some smooth function $K(p,s)$. Substituting this expression for G in the definition of $\mathcal{N}^*$ the smoothness of $\mathcal{N}^*G$ becomes clear.

Also

$$\begin{aligned}
\langle \mathcal{D}G_1\,, G_2\rangle &= \int_{|p|=\rho}\int_0^\infty \left(\frac{1}{2t}\frac{\partial}{\partial t}\right)^{(n-3)/2} (G_1(p,t))\; G_2(p,t)\,dt\,dS_p \\
&= (-1)^{(n-3)/2}\int_{|p|=\rho}\int_0^\infty G_1(p,t)\;\left(\frac{\partial}{\partial t}\frac{1}{2t}\right)^{(n-3)/2} (G_2(p,t))\,dt\,dS_p \\
&= (-1)^{(n-3)/2}\int_{|p|=\rho}\int_0^\infty G_1(p,t)\,t\;\left(\frac{1}{2t}\frac{\partial}{\partial t}\right)^{(n-3)/2} \left(\frac{G_2(p,t)}{t}\right)\,dt\,dS_p \\
&= \langle G_1\,, \mathcal{D}^* G_2\rangle
\end{aligned}$$

if we take

$$(\mathcal{D}^*G)(p,t) = (-1)^{(n-3)/2} t\,\mathcal{D}(G(p,t)/t)\ . \tag{2}$$

Note that $\mathcal{D}^*$ maps functions in $\tilde{C}^\infty(S_\rho(0)\times[0,\infty))$ to functions in $\tilde{C}^\infty(S_\rho(0)\times[0,\infty))$.

Theorem 3 (Inversion Formula). *If $n\geq 3$ and odd, f is a smooth function supported in $\overline{B_\rho(0)}$ and $(Mf)(p,r)$ (and hence $(\mathcal{N}f)(p,r)$) is known for all $p\in S_\rho(0)$ and all real r, then we have the explicit inversion formulas*

$$\begin{aligned}
f(x) &= -\frac{\pi}{2\;\rho\,\Gamma(n/2)^2}\,\left(\mathcal{N}^*\,\mathcal{D}^*\,\partial_t^2\,t\,\mathcal{D}\,\mathcal{N}f\right)(x), \qquad x\in B_\rho(0)\\
f(x) &= -\frac{\pi}{2\,\rho\,\Gamma(n/2)^2}\,(\mathcal{N}^*\,\mathcal{D}^*\,\partial_t\,t\,\partial_t\,\mathcal{D}\,\mathcal{N}f)(x), \qquad x\in B_\rho(0)\\
f(x) &= -\frac{\pi}{2\,\rho\,\Gamma(n/2)^2}\,\Delta_x\,(\mathcal{N}^*\,\mathcal{D}^*\,t\,\mathcal{D}\,\mathcal{N}f)(x), \qquad x\in B_\rho(0)\ .
\end{aligned}$$

The inversion formulas in Theorem 3 are local in the sense that $f(x)$ is determined purely from the mean values of f over spheres centered on $S_\rho(0)$ passing through an arbitrarily small neighborhood of x. These inversion formulas also generate energy L^2 norm identities which are a step towards a characterization of the range of the map $f\to Mf$.

There is some similarity between the inversion formula in Theorem 3 and the inversion formula for the Radon transform. The Radon transform of a function f on R^n is

$$(\mathcal{R}f)(\theta,r) = \int_{x\cdot\theta=r} f(x)\,dS_x, \qquad \forall r\in(-\infty,\infty),\ \theta\in R^n,\ |\theta|=1.$$

Its L^2 adjoint is, for every function F on $S_1(0)\times(-\infty,\infty)$,

$$(\mathcal{R}^*F)(x) = \int_{|\theta|=1} F(\theta, x\cdot\theta)\,d\theta, \qquad \forall x\in R^n,$$

and the inversion formula for the Radon Transform is (see [25])

$$f(x) = \frac{(-1)^{(n-1)/2}}{2(2\pi)^{n-1}}\Delta_x^{(n-1)/2}(\mathcal{R}^*\mathcal{R}f)(x), \qquad \forall x\in R^n\ .$$

The above theorems will be proved by converting the problem to a problem about the solutions of the wave equation. Consider the initial value problem

$$\Box u \equiv u_{tt} - \Delta u = 0, \qquad x \in R^n,\ t \in R \tag{3}$$

$$u(.,t{=}0) = 0,\ u_t(.,t{=}0) = f(.), \tag{4}$$

with f smooth and supported in $\overline{D}$. Then, from the standard theory for solutions of the wave equation, u is smooth in x, t, odd in t (because $-u(x,-t)$ is also the solution), and as shown in [7], page 682, for $n \geq 2$,

$$u(x,t) = \frac{1}{(n-2)!}\frac{\partial^{n-2}}{\partial t^{n-2}} \int_0^t r\,(t^2 - r^2)^{(n-3)/2}\,(Mf)(x,r)\,dr, \qquad t \geq 0\ . \tag{5}$$

Hence the original problem is equivalent to the problem of recovering $u_t(x,0)$ from the value of $u(x,t)$ on subsets of $S \times (-\infty, \infty)$. So Theorems 1, 2 will follow from the following theorems.

Theorem 4 (Uniqueness). *Suppose D is a bounded, open, subset of R^n, $n \geq 2$, with a smooth boundary S, and $\overline{D}$ is strictly convex. Let Γ be a relatively open subset of S. Suppose f is a smooth function on R^n, supported in $\overline{D}$, and u is the solution of the initial value problem (3), (4). If $u(p,t) = 0$ for all $p \in \Gamma$ and all t then $f = 0$.*

The appropriate version of this result for $n = 1$ is also true and may be shown by arguments similar to (but simpler than) those used in proving the above theorem.

Theorem 5 (Reconstruction). *Suppose D is a bounded, open, connected subset of R^n, n odd, with a smooth boundary S. Suppose f is a smooth function on R^n, supported in $\overline{D}$, and u is the solution of the IVP (3), (4). If $u(p,t)$ is known for all p in S and for all $t \in [0, diam(D)/2]$ then we may recover f. We have a simpler algorithm if $u(p,t)$ is known for all $t \in R$ (and all $p \in S$).*

In our reconstruction procedures we use the fact that for n odd the fundamental solution of the wave operator is supported on the cone $t^2 = |x|^2$. This is not true in even dimensions and so our algorithm is not valid in even space dimensions. Further, the method of descent does not help, because if we consider u as a function of an additional one dimensional variable z, of which u is independent, then the initial data of the new u is supported in an infinite cylinder in x, z space, and hence is not supported in a bounded domain.

We show, at the end of the introduction, that Theorem 3 follows from

Theorem 6 (Trace Identity). *Suppose $n \geq 3$, n odd, $\rho > 0$, $f_i \in C_0^\infty(\overline{B_\rho(0)})$, and u_i is the solution of the IVP (3), (4) for $f = f_i$, $i = 1, 2$. Then we have the identities*

$$\frac{1}{2}\int_{R^n} f_1(x)\,f_2(x)\,dx \;=\; \frac{-1}{\rho}\int_0^\infty \int_{|p|=\rho} t\,u_1(p,t)\ u_{2tt}(p,t)\,dS_p\,dt, \tag{6}$$

$$\frac{1}{2}\int_{R^n} f_1(x)\,f_2(x)\,dx \;=\; \frac{1}{\rho}\int_0^\infty \int_{|p|=\rho} t\,u_{1t}(p,t)\ u_{2t}(p,t)\,dS_p\,dt. \tag{7}$$

Note that (6) is not symmetric so it clearly implies another similar identity. Some other interesting consequences of the non-symmetry will be addressed elsewhere.

We do not have an inversion formula similar to the one in Theorem 3 or Theorem 6 for even dimensions. If we can prove an inversion formula or an identity of the above type for the $n = 2$ case, even when f is spherically symmetric, then we feel that the techniques used in the proof of Theorem 6 would carry over to a proof for all n even and all f (not just spherically symmetric f). However, we do not have an inversion formula in the $n = 2$ case even when f is spherically symmetric.

The identity in Theorem 3 is a step towards identifying the range of the map $f \to (Mf)(p, r)$ in the case when D is a ball in R^n, n odd. The other theorems do not attempt to specify the range of this map for the general case. The identity in Theorem 6 has important implications for optimal regularity of traces of solutions of hyperbolic partial differential equations whose principal part is the wave operator. Some of this may be seen in the proof of Theorem 6 but the general trace regularity results and their proofs will be given in [11].

The recovery of a function from its mean values over spheres centered on some surface or other families of surfaces has been studied by many authors. John [15] is a good source for the early work on recovering a function from its mean values over a family of spheres with centers on a plane. A very interesting theoretical analysis of the problem with centers restricted to a plane was provided by Bukhgeim and Kardakov in [5]; see also the work of Fawcett [9] and Andersson [4] for additional results for this problem. The difficult problem of recovering a function from integrals over a fairly general family of surfaces has also been studied - see [22] and [23] and the references there. The results in our article, for the very specialized family of surfaces we consider, are stronger.

Cormack and Quinto in [6] and Yagle in [35] studied the recovery of f from the mean values of f over spheres passing through a fixed point. Volchkov in [31] studied the injectivity issue in the problem of recovering a function from its mean values over a family of spheres. He characterizes injectivity sets which have a spherical symmetry so these results do not cover the injectivity result in Theorem 1. Using techniques from $\mathcal{D}$-module theory, Goncharov in [13] finds explicit inversion formulas for the spherical mean value transform operator restricted to some n-dimensional varieties of spheres in R^n. The variety of spheres tangent to a hypersurface is included, but our interest, the family of spheres centered on a hypersurface, is not.

Agranovsky and Quinto in [1], [2] have proved several significant uniqueness results for the spherical mean transform, and applied them to related questions such as stationary sets for solutions of the wave equation. In [1] they give a complete characterization of sets of uniqueness (sets of centers) for the spherical mean transform on compactly supported functions in the plane, i.e. without assumption on the location of the support with respect to the set of centers. In [21] there is an announcement of a uniqueness theorem more general than our Theorem 1, which can be proved using techniques from microlocal analysis in the analytic category as exposed in section 3 of [2]. We think that our proof is still interesting. We use domain of dependence arguments and unique continuation for the time-like Cauchy problem to prove Theorem 4, and hence Theorem 1. Since the domain of dependence result and the unique continuation result for the time-like Cauchy problem are valid for very general hyperbolic operators (with coefficients independent of t), our proof of Theorem 4 is actually valid if the wave operator is replaced by a first order perturbation with coefficients which are C^1 and independent of t. Our technique may perhaps extend to solutions of more general hyperbolic operators with non-constant reasonably smooth coefficients, whereas the methods in [2], [21] would carry over, at most, to operators with analytic coefficients.

Theorem 3 in [3] also addresses a question similar to the one dealt in Theorem 1. There, they are interested in the uniqueness question when the mean values of f are known for all spheres centered on the boundary but they do not require that f be supported inside the region D. They show uniqueness holds if $f \in L^q(R^n)$ as long as $q \leq 2n/(n-1)$. The theorem fails for $q > 2n/(n-1)$.

Norton in [26] derived an explicit inversion formula for the $n = 2$ case, of the the problem discussed in Theorem 3, using an expansion in Bessel functions. The inversion formula needs further analysis to analyze the effect of the zeros of Bessel functions used in the formula. Norton and Linzer in [27] considered the recovery of f (supported in a ball in R^3) from the mean values of f over all spheres centered on the boundary of the ball, again by using a harmonic decomposition. They also related it to the solution of the wave equation and then transferred the problem to the frequency domain by taking the Fourier Transform of the time variable. There they provide an inversion formula in the form of an integral operator whose kernel is given by an infinite sum. Then they truncated this sum to obtain an approximate inversion formula. They did not deal with the higher dimensional case. Our exact inversion formula, which is valid in all odd dimensions, seems to have a cleaner closed form. The recent articles [32]-[34] use the work of Norton and Linzer, [27], for reconstruction in thermoacoustic tomography.

After the presentation of some of the results of this paper at Oberwolfach, A G Ramm informed us that he could also invert the spherical mean transform with centers on some surfaces, and sent us the preprint [28]. For the problem of inversion when centers lie on a sphere he gives a series method whose details are given for dimension $n = 3$. In that case, his result can already be found in formulas (52) and (56) of Norton and Linzer in [27]. He also establishes a uniqueness theorem whose strength in relation to prior results is not fully clear, but it does not contain our Theorem 1, for example.

We conclude the introduction by showing how Theorem 3 follows from Theorem 6. For n odd, $n \geq 3$, from page 682 of [7], we have a more convenient representation of $u(x,t)$ in terms of $(Mf)(x,r)$ than the one given earlier. We have

$$u(x,t) = \frac{\sqrt{\pi}}{2\,\Gamma(n/2)} \left(\frac{1}{2t}\frac{\partial}{\partial t}\right)^{(n-3)/2} \left(t^{n-2}(Mf)(x,t)\right) = \frac{\sqrt{\pi}}{2\,\Gamma(n/2)} \mathcal{D}\,\mathcal{N}(f)(x,t)\ . \tag{8}$$

Hence, for all $f_1, f_2 \in C_0^\infty(\overline{B_\rho(0)})$, (6) and (7) may be rewritten as

$$\langle f_1, f_2 \rangle = \frac{-\pi}{2\,\rho\,\Gamma(n/2)^2}\,\langle t\,\mathcal{DN} f_1\,,\,\partial_t^2 \mathcal{DN} f_2\rangle = \frac{-\pi}{2\,\rho\,\Gamma(n/2)^2}\,\langle \mathcal{N}^*\,\mathcal{D}^*\,\partial_t^2\,t\,\mathcal{D}\,\mathcal{N} f_1\,,\,f_2\rangle,$$

$$\langle f_1, f_2 \rangle = \frac{\pi}{2\,\rho\,\Gamma(n/2)^2}\,\langle t\,\partial_t \mathcal{DN} f_1\,,\,\partial_t \mathcal{DN} f_2\rangle = \frac{-\pi}{2\,\rho\,\Gamma(n/2)^2}\,\langle \mathcal{N}^*\,\mathcal{D}^*\,\partial_t\,t\,\partial_t\,\mathcal{D}\,\mathcal{N} f_1\,,\,f_2\rangle\,.$$

We have an additional identity which comes from the observation that if u is a solution of (3), (4), then u_{tt} is also a solution of (3) but with the ICs $u_{tt}(.,t{=}0) = 0$ and $u_{ttt}(.,t{=}0) = \Delta f$. Hence (6) also implies

$$\langle f_1, f_2 \rangle = \frac{-\pi}{2\,\rho\,\Gamma(n/2)^2}\,\langle t\,\mathcal{DN} f_1\,,\,\mathcal{DN}\Delta f_2\rangle = \frac{-\pi}{2\,\rho\,\Gamma(n/2)^2}\,\langle \Delta\mathcal{N}^*\,\mathcal{D}^*\,t\,\mathcal{D}\,\mathcal{N} f_1\,,\,f_2\rangle\,.$$

These give us the three inversion formulas of Theorem 3.

2. Proof Of Theorem 4

We will need three results in the proof of Theorem 4.

2.1. Unique Continuation For Time Like Surfaces.

The first result concerns unique continuation for the time-like Cauchy problem for the wave equation.

Proposition 1. *If u is a distribution and satisfies (3) and u is zero on $B_\epsilon(p)\times(-T,T)$ for some $\epsilon > 0$, and $p \in R^n$, then u is zero on*

$$\{\ (x,t)\ :\ |x-p| + |t| < T\ \},$$

and in particular on

$$\{\ (x,t{=}0)\ :\ |x-p| < T\ \}\ .$$

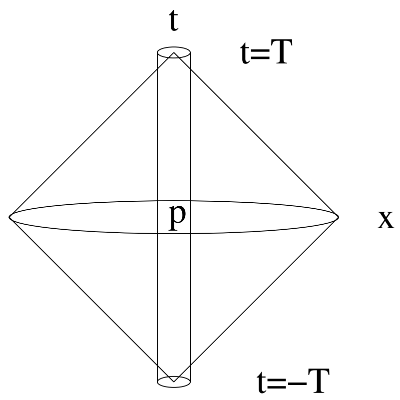


Figure 2. John's Theorem

Proof of Proposition 1

Proposition 1 follows quickly from Theorem 8.6.8 in [14], which itself is derived from Holmgren's theorem. In that theorem take $X_2 = \{\ (x,t)\ :\ |x-p| + |t| < T\ \}$ and $X_1 = X_2 \cap (B_\epsilon(p) \times (-T,T))$, and note that any characteristic hyperplane through a point in X_2 has the form $(x - x_0)\cdot\theta + (t - t_0) = 0$ for some unit vector θ and some point $(x_0, t_0) \in X_2$. This plane cuts the vertical line $x = p$ (in (x, t) space) at the point (p, t) where $t = t_0 - (p - x_0) \cdot \theta$ and hence $|t| \leq |t_0| + |p - x_0| < T$. QED

While Proposition 1 is well known in certain circles, we did not find a ready reference for the proof and so have included the proof above. The proposition was generalized by Robbiano and Hormander to apply to hyperbolic operators with coefficients independent of t, but the generalization was not as sharp as Proposition 1. The definitive form, due to Tataru in [30], includes Proposition 1 as a special case. The proof of Tataru's result is quite complicated, but for Theorem 4 we need only the special case above. The possible extension of Theorem 4 to more general hyperbolic operators would require the full strength of Tataru's result.

2.2. **Domain Of Dependence For Exterior Problems.** Let D be a bounded, open, subset of R^n with a smooth boundary S. For points p, q outside D, let $d(p, q)$ denote the infimum of the lengths of all the piecewise C^1 paths in $R^n \setminus D$ joining p to q. Using ideas in Chapter 6 of [24] (where it is applied to the distance function generated by a Riemannian metric), one may show that $d(p, q)$ is a topological metric on $R^n \setminus D$.

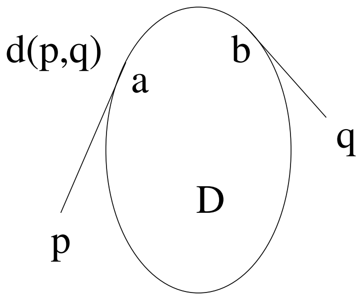


FIGURE 3. Shortest path between p and q

For any point p in $R^n \setminus D$ and any positive number r, define $E_r(p)$ to be the p centered ball of radius r in $R^n \setminus D$ under this metric, that is

$$E_r(p) = \{\, x \in R^n \setminus D \;:\; d(x, p) < r \,\} .$$

The second result we need in the proof of Theorem 4 is about the domain of dependence of solutions of the wave equation on an exterior domain. Loosely speaking, the result claims that the value, of the solution of the wave equation in an exterior domain, at a point (x, s), affects the value of the solution at the point (y, t) only if $d(x, y) \leq t - s$.

Proposition 2 (Domain of Dependence). *Suppose D is a bounded, connected, open subset of R^n with a smooth boundary S. Suppose u is a smooth solution of the exterior problem*

$$u_{tt} - \Delta u = 0, \qquad x \in R^n \setminus D,\ t \in R$$

$$u = h \qquad on\ S \times R\ .$$

Suppose p is not in D, and $t_0 < t_1$ are real numbers. If $u(.,t_0)$ and $u_t(.,t_0)$ are zero on $E_{t_1-t_0}(p)$ and h is zero on

$$\{(x,t) \;:\; x \in S,\; t_0 \leq t \leq t_1,\; d(x,p) \leq t_1 - t\,\},$$

then $u(p,t)$ and $u_t(p,t)$ are zero for all $t \in [t_0, t_1)$.

In textbooks one may find proofs of this result when $D = \emptyset$ (in which case $d(x,y) = |x-y|$), whereas we are interested in the result for solutions in exterior domains. While the method of attack for proving such a result is clear enough, the details are complicated by the fact that the map $x \to d(x,p)$ is not a smooth map and hence one has to appeal to a more general version of the Divergence Theorem.

To prove Proposition 2 we first show that for any p outside D, the function $x \to d(x,p)$ is a locally Lipschitz function on $R^n \setminus D$; that is for every point $q \in R^n \setminus D$, there is a ball $B_\rho(q)$ such that

$$|d(x,p) - d(y,p)| \leq C|x-y|, \qquad \forall x, y \in B_\rho(q) \setminus D$$

with C independent of x and y. From the triangle inequality

$$|d(x,p) - d(y,p)| \leq d(x,y)$$

so the Lipschitz nature will follow if we can show that, for every $q \in R^n \setminus D$, there is a ball $B_\rho(q)$ such that

$$d(x,y) \leq C|x-y|, \qquad \forall\; x, y \in B_\rho(q) \setminus D, \tag{9}$$

with C independent of x, y. We give the proof below.

If q is not on the boundary of D, then we can find a ball $B_\rho(q)$ contained in $R^n \setminus D$ and hence $d(x,y) = |x-y|$ for all $x, y \in B_\rho(q)$. So the challenge is to prove (9) for $q \in S$. We give a proof of (9) in the $n = 3$ case - the general case is very similar just the notation gets a little cumbersome. For $q \in S$, without loss of generality, we may find a ball $B_\rho(q)$ so that

$$D \cap B_\rho(q) = \{u = (u_1, u_2, u_3) \;:\; u_3 > \phi(u_1, u_2),\; u \in B_\rho(q)\}$$

for some smooth function $\phi(u_1, u_2)$. Now if $x, y \in B_\rho(q) \setminus D$ then $x_3 \leq \phi(x_1, x_2)$, $y_3 \leq \phi(y_1, y_2)$. If the line segment xy does not intersect D then $d(x,y) = |x-y|$ and (9) is valid. So assume that the segment xy enters $\overline{D}$ at a and leaves $\overline{D}$ the last time at b. Then

$$d(x,y) \leq |x-a| + d(a,b) + |b-y| \leq |x-y| + d(a,b) + |x-y| = 2|x-y| + d(a,b)\;.$$

So if we could prove $d(a,b) \leq C|a-b|$ for a, b on the boundary of D then (9) would follow because $|a-b| \leq |x-y|$. So let us estimate $d(a,b)$. From its definition, $d(a,b)$ is not larger than the length of the projection of the line segment ab onto S. Now the projection of the segment ab onto S is

$$s \to \mathbf{r}(s) = (1-s)[a_1, a_2, 0] + s[b_1, b_2, 0] + [0, 0, \phi((1-s)a_1 + sb_1, (1-s)a_2 + sb_2)] \qquad 0 \leq s \leq 1\;.$$

Hence

$$|\mathbf{r}'(s)| = |\,[b_1 - a_1, b_2 - a_2, \phi_1(.,.)(b_1 - a_1) + \phi_2(.,.)(b_2 - a_2)]\,| \leq C|b-a|$$

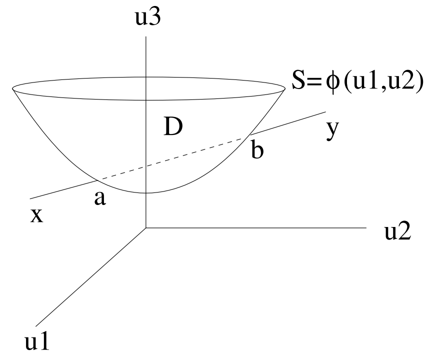


FIGURE 4. Lipschitz Estimate

because the partial derivatives of ϕ are bounded on S. Hence the length of the projection is no more than $C|b-a|$. This proves (9) for $x, y \in B_\rho(q)$.

Since the map $x \to d(x,p)$ is Lipschitz, from Rademacher's theorem (see [8]), $d(x,p)$ is differentiable almost everywhere in $R^n \setminus D$. Let us estimate $|\nabla_x d(x,p)|$ (if it exists) for $x \notin \overline{D}$. For x not in $\overline{D}$, there is a ball around x which does not intersect $\overline{D}$, and hence for any y in this ball we have $d(x,y) = |x-y|$. Since $d(x,p)$ is a metric, for any y in this ball $|d(y,p) - d(x,p)| \leq d(x,y) = |x-y|$ and hence

$$\frac{|d(y,p) - d(x,p)|}{|y-x|} \leq 1$$

for all y in the ball. Hence the directional derivative of $d(x,p)$, at x, in any direction, does not exceed 1, and hence $|\nabla_x d(x,p)| \leq 1$, for all x not in $\overline{D}$, where it exists. Actually, we believe $|\nabla d(x,p)| = 1$ almost everywhere but we do not need this.

Proof of Proposition 2

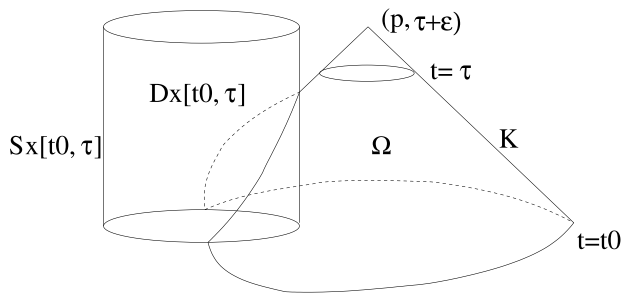


FIGURE 5. Domain Of Integration

For any real number τ in (t_0, t_1), choose $\epsilon > 0$ so that $\tau + \epsilon < t_1$. Let $\mathcal{K}$ be the backward "conical" surface $t = \tau + \epsilon - d(x,p)$ in x, t space with vertex $(p, \tau+\epsilon)$, defined by $d(p,q)$. Specifically

$$\mathcal{K} = \{ \ (x, \tau + \epsilon - d(x,p)) \ : \ x \in R^n \setminus D \ \} \ .$$

Since $d(x,p)$ is a Lipschitz function, $\mathcal{K}$ is a Lipschitz surface and so has a normal almost everywhere. For points of K corresponding to x not in S, the upward pointing normal (where it exists) will be parallel to $(\nabla_x d(x,p),1)$. So if (ν_x,ν_t) is the upward pointing unit normal to $\mathcal{K}$ then

$$|\nu_x| = \frac{|\nabla_x d(x,p)|}{\sqrt{1+|\nabla_x d(x,p)|^2}} \leq \frac{1}{\sqrt{1+|\nabla_x d(x,p)|^2}} = \nu_t \ .$$

To prove the domain of dependence result we imitate the proof used for such a result if the domain were the whole space. We will perform an integration over the region Ω (which is the subset of $(R^n \setminus D) \times R$) enclosed by the planes $t = \tau$, $t = t_0$, the surface $S \times [t_0, \tau]$, and the backward cone $\mathcal{K}$. Since Ω need not be a domain with a smooth (or even C^1) boundary we will appeal to a generalization of the divergence theorem - the generalized Gauss-Green theorem of Federer. Please see the appendix and [8] for definitions and the statement of the results below. Let Φ be the n-dimensional Hausdorff measure on R^{n+1} (which is a regular Borel measure on R^{n+1}), and $(\nu_x,\nu_t) = \nu = \nu(\Omega,x,t)$ be the generalized outward pointing unit normal at (x,t) associated with the region Ω.

We have

$$(u_t^2 + |\nabla u|^2)_t - 2\nabla\cdot(u_t\nabla u) = 2u_t(u_{tt} - \Delta u) = 0 \qquad \text{in } \Omega \ .$$

Hence from the Gauss-Green theorem (Proposition 6 in the Appendix)

$$0 = \int_\Omega (u_t^2 + |\nabla u|^2)\nu_t - 2u_t \nabla u \cdot \nu_x \ \ d\Phi \ . \tag{10}$$

Note that to apply the Gauss-Green theorem we must make sure that the Φ measure of $\partial\Omega$ is finite. But $\partial\Omega$ is a subset of the union of bounded parts of the surfaces of $t = t_0$, $t = \tau$, $\mathcal{K}$ and $S \times [t_0, \tau]$ and the Φ measure of these surfaces equals their surface area (for Lipschitz surfaces) and all these surface areas are finite (including $\mathcal{K} \, : \, t = \tau + \epsilon - d(x,p)$ because $|\nabla_x d(x,p)| \leq 1$). Note that all the sets entering our discussion are Borel sets.

Now, from the definition - see the Appendix, $\nu(x,t) = 0$ at all interior points of Ω and Ω^c. So we need $\nu(\Omega,x,t)$ for points (x,t) on the boundary of Ω. The boundary of Ω consists of a part coming from $t = \tau$, a part coming from $t = \tau_0$, a part from $\mathcal{K}$, and a part from $S \times [t_0,\tau]$. The generalized normal agrees with the usual normal to surfaces at points where the boundary is Lipschitz (so where it is smooth). The boundary is Lipschitz at all those points which lie on only one of the bounding surfaces - the difficulties arise at points where one or more bounding surfaces of Ω meet. Hence at most points of $\partial\Omega$, on $t = \tau$ we have $\nu = (0,1)$, on $t = t_0$ we have $\nu = (0,-1)$, on $S \times [t_0,\tau]$ we have $\nu = (\nu_x, 0)$, and on $\mathcal{K}$ we have $\nu = (\nu_x,\nu_t)$ with $|\nu_x| \leq \nu_t$.

Now we must deal with the boundary points which lie on the intersection of two or more of the bounding surfaces of Ω. If we could prove that the Φ measure of this set is zero then we would not need to know the value of $\nu(\Omega,x,t)$ for points on this set. This is true perhaps if all surfaces were C^1 but we are not sure if this is true if one them is Lipschitz. So we must determine $\nu(\Omega,x,t)$ at these special points on the boundary. From Proposition 5 in the Appendix, if the special point lies at the intersection of (two smooth non-tangential) surfaces $t = t_0$ or $t = \tau$ with $S \times [t_0,\tau]$

then $\nu = 0$ at that point; if the special point lies at the intersection of $\mathcal{K}$ with $t = t_0$ or $t = \tau$ or $S \times [t_0, \tau]$ then either $\nu = 0$ at that point or ν is the normal at that point to the corresponding smooth surface $t = t_0$ or $t = \tau$ or $S \times [t_0, \tau]$ (as if $\mathcal{K}$ did not play a part).

Now we examine the contribution to the RHS of (10) from the various parts. Based on our description of $\nu(\Omega, x, t)$ in the previous paragraph, we get non-zero contributions, at most, from points on the boundary of Ω. The contribution from the $S \times [t_0, \tau]$ part will be zero because $\nu_t = 0$ on $S \times [t_0, \tau]$ and u and hence u_t is zero on the part of $\partial\Omega$ on $S \times [t_0, \tau]$. The contribution from the $t = \tau_0$ parts is zero because u and u_t are zero on the part of $\partial\Omega$ on $t = t_0$. The contribution from the $\mathcal{K}$ part of $\partial\Omega$ which is not on any of the other parts, is non-negative because for this part $\nu_t(x,t) \geq |\nu_x(x,t)| \geq 0$ for $x \notin \overline{D}$, and hence the integrand is non-negative because

$$\begin{aligned} (u_t^2 + |\nabla u|^2)\nu_t - 2u_t \nabla u \cdot \nu_x &\geq (u_t^2 + |\nabla u|^2)\nu_t - 2|u_t||\nabla u||\nu_x| \\ &\geq \nu_t(u_t^2 + |\nabla u|^2 - 2|u_t||\nabla u|) = \nu_t(|u_t| - |\nabla u|)^2 \ . \end{aligned}$$

Hence the contribution from the $t = \tau$ part (which is non-negative because $\nu_x = 0$ and $\nu_t = 1$ on $t = \tau$) must be zero. Further the integration is over a region lying above the part of $B_\epsilon(p)$ outside D. Hence $u_t(p, \tau) = 0$ for every $\tau \in [t_0, t_1)$. Also $u(p, t_0) = 0$ by hypothesis, hence $u(p, \tau) = 0$ for all $\tau \in [t_0, t_1)$. **QED**

2.3. **Distance Computation.** The third intermediate result we need is the crucial computation of a certain distance. Below, when we refer to the boundary or the closure of $E_r(p)$ we mean that as a subset of R^n in the topology of R^n and not in the topology induced by $d(p,q)$.

Proposition 3. *Suppose D is a bounded, open subset of R^n, $n \geq 2$, S is its smooth boundary, and $\overline{D}$ is strictly convex. Suppose p is a point on S and ρ a small enough positive number less than r. If $K = \overline{D} \setminus B_r(p)$ is not empty then the shortest distance between K and the closure of $E_\rho(p)$ is the length of some line segment joining a point on $S \cap S_r(p)$ to a point on the boundary of $E_\rho(p)$ which lies on S.*

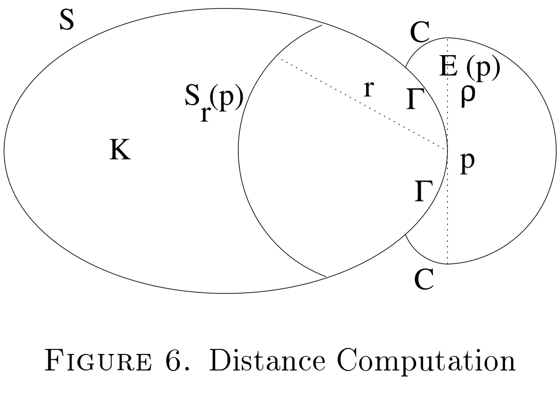


FIGURE 6. Distance Computation

Proof of Proposition 3
Let δ be the shortest distance between K and the closure of $E_\rho(p)$. Let Γ be the subset of S consisting of points whose geodesic distance from p (on S) is less than

or equal to ρ. The boundary of K consists of a part on S which we call the outer boundary and the rest (which is on $S_r(p)$) which we call the inner boundary. The boundary of $E_\rho(p)$ consists of a part to the right of the tangent plane to S at p (the part in the region $(x-p)\cdot\nu_p > 0$ where ν_p is the exterior normal to S at p), a part on S which is Γ, and the rest which we denote by C.

It is clear enough that the shortest distance will be the distance between some point on the boundary of K and some point on the boundary of $E_\rho(p)$. Further, a shortest segment will be normal to the two boundaries provided the boundaries are smooth at the optimal points.

Because of the strict convexity of $\overline{D}$ we can find points p' on S arbitrarily close to p so that the distance between p' and the inner boundary of K is less than r, implying $\delta < r$. In fact pick a point q in the interior of the inner boundary of K so that the segment pq is not normal to S. Then we can find a direction tangential to S so that if p moves in that direction, on S, then $|p-q|$ will decrease. So an optimal point on ∂K can not be on the interior of the inner boundary of K, else an optimal segment will be normal to $S_r(p)$ and hence would pass through p.

An optimal point, on the boundary of $E_\rho(p)$, can not be to the right of the tangent plane to S at p because then the corresponding optimal line will be normal to $S_\rho(p)$, and hence will pass through p, and then p will be a better candidate than this point.

Next we claim that no point of C is a candidate for an optimal point on the boundary of $E_\rho(p)$, unless it is on Γ. We show this by showing that for any point q on $S\setminus\Gamma$, the point on C closest to q is on Γ.

For ρ small enough, we may parameterize Γ by (s,θ) where s is the geodesic distance from p and θ is a unit vector representing the tangent vector to the geodesic at p. So the surface Γ is

$$x = \gamma(s,\theta), \qquad 0 \le s \le \rho,\ |\theta| = 1,\ \theta \in R^{n-1}$$

and for each fixed θ the curve $s \to \gamma(s,\theta)$ with $s \in [0,\rho]$ is a geodesic on S and s is the arc length along this geodesic. So $\gamma_{ss}(s,.)$ is normal to S at $\gamma(s,.)$ and $|\gamma_s| = 1$.

Further, because $\overline{D}$ is strictly convex, for ρ small enough, for any point q in the closure of $E_\rho(p)$, the $d(p,q)$ is attained either as the length of the segment pq or the length of a curve consisting of a geodesic on S, starting at p, followed by a line segment from the end point of the geodesic to q which is tangential to the geodesic - see Figure 3 (of course p is on S in our case). So, for ρ small enough, C is generated by the family of curves

$$s \to c(s,\theta) = \gamma(s,\theta) + (\rho - s)\gamma_s(s,\theta), \qquad 0 \le s \le \rho$$

as θ ranges over the unit sphere in R^{n-1}.

Let us examine the distance between q and points on one of the generating curves of C. Define $h(s) = |c(s,.) - q|^2$ - the square of the distance between q and a point on a generating curve. Then for $0 \le s \le \rho$

$$\begin{aligned} h'(s) &= 2(c(s,.) - q)\cdot c_s(s,.) \\ &= 2(\gamma(s,.) + (\rho - s)\gamma_s(s,.) - q)\cdot\gamma_{ss}(s,.)(\rho - s) \\ &= 2(\rho - s)(\gamma(s,.) - q)\cdot\gamma_{ss}(s,.)\ . \end{aligned}$$

Above we used $\gamma_s \cdot \gamma_{ss} = 0$ because γ_s is tangential to S and γ_{ss} is normal to S. Now $\gamma(s,.)$ is a point on S (actually on Γ) - denote it by a, and $\gamma_{ss}(s,.)$ is the inward pointing normal to S there. Hence the strict convexity of $\overline{D}$ implies (note $q \neq a$)

$$0 < (q-a) \cdot \gamma_{ss} = (q - \gamma(s,.)) \cdot \gamma_{ss}(s,.) \ .$$

Hence $h'(s) < 0$ for $0 \leq s < \rho$ and so $h(s)$, on $0 \leq s \leq \rho$, attains its minimum at $s = \rho$, that is at the point $c(\rho,.)$, which is $\gamma(\rho,.)$, which lies on Γ. This proves that the point on C closest to a fixed point q on $S \setminus \Gamma$ must be on Γ.

Because D is strictly convex, the normal lines to the exterior boundary of K, will have to cross the inner boundary of K before they meet Γ. To see this suppose the normal line connects a point x on the exterior boundary to a point y on Γ. Then strict convexity of $\overline{D}$ implies that the line segment xy is in D (except for the end points). Now $|x-p| > r$ and $|y-p| < r$ so there is a point z on the line segment xy so that $|z-p| = r$ and hence z is on the inner boundary of K. Hence no interior point of the outer boundary of K can be an optimal point. This completes the proof of Proposition 3.

2.4. **Proof of Theorem 4.** We now give the proof of Theorem 4. Without loss of generality we may assume that there is a point p on S and a small positive real number ρ so that $\Gamma = \overline{E_\rho(p)} \cap S$.

Step 1

Choose any $\epsilon > 0$ smaller than ρ. Let q be any point in the hemisphere $H \cap B_\epsilon(p)$ where H is the the region to the right of, and includes, the tangent plane to S at p (see Figure 6); so H is the half-space in x-space containing p and not intersecting D. Then $d(p,q) = |p-q| < \epsilon$ and hence from the triangle inequality, for any x in $S \setminus \Gamma$, we have $d(x,q) \geq d(x,p) - d(p,q) \geq \rho - \epsilon$.

Let $u = h$ on $S \times R$. Then u is the solution of the exterior problem

$$\begin{aligned} u_{tt} - \Delta u = 0, \qquad & x \in R^n \setminus D, \ t \in R \\ u(x,t{=}0) = 0, \ u_t(x,t{=}0) = 0, \qquad & x \in R^n \setminus D \\ u = h \qquad & \text{on } S \times R \ . \end{aligned}$$

Now h is supported in $(S \setminus \Gamma) \times R$ and the distance of q from $S \setminus \Gamma$ is at least $\rho - \epsilon$. Hence from Proposition 2 we have $u(q,t)$ is zero for $|t| < \rho - \epsilon$, for all $q \in H \cap B_\epsilon(p)$.

Now u is the solution of the wave equation on $R^n \times R$, so the previous result combined with Proposition 1 gives that $f(x) = u_t(x,t{=}0)$ is zero on

$$\{ \, x \in R^n \ : \ |x - q^*| < \rho - \epsilon \, \}$$

for some (actually all) q^* in the interior of $H \cap B_\epsilon(p)$, for all small $\epsilon > 0$. Since $|x - q^*| \leq |x-p| + |p - q^*|$ so $f(x)$ is zero on

$$\{ \, x \in R^n \ : \ |x - p| < \rho - 2\epsilon \, \}$$

for all $\epsilon > 0$. Hence $f(x)$ is zero on

$$\{ \, x \in R^n \ : \ |x - p| < \rho \, \} \ .$$

Step 2

We now show that $f(x)$ is zero for all x. This will follow easily if we can show the following - if $f(x)$ is zero on the region $|x-p|<r$ for some $r \geq \rho$ then $f(x)$ is zero on the region $|x-p|<r+\sigma$ where σ is a positive number independent of r.

Please refer to Figure 6 for a geometrical interpretation of the notation below. So suppose f is supported in the region K consisting of the part of $\overline{D}$ outside $B_r(p)$. Let $\delta>0$ be the straight line distance between $E_\rho(p)$ and K then we show that f is zero on $B_{\rho+\delta}(p)$. Postponing the proof of this claim, let α be the supremum of the straight line distances between p and points on Γ. Since $\overline{D}$ is strictly convex, from the definition of Γ, we have $\alpha<\rho$. From Proposition 3, δ is the length of the line

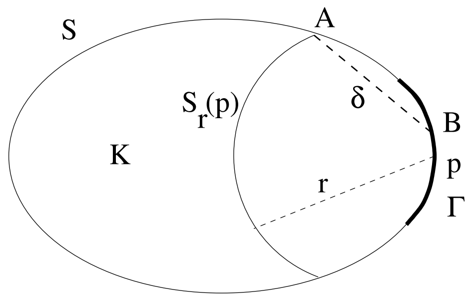


FIGURE 7. Triangle Inequality

segment AB for some point A on $S_r(p) \cap S$ and some point B on Γ. Then, using the triangle inequality,

$$\begin{aligned} \rho+\delta &= \rho+|AB| = |AB|+|Bp|+(\rho-|Bp|) \\ &\geq |pA|+(\rho-|Bp|) \geq r+(\rho-\alpha) \end{aligned}$$

and note that $\rho-\alpha$ is positive and independent of r. Hence Theorem 4 holds.

So it remains to show that if f is supported in $K=\overline{D}-B_r(p)$ then f is zero on $B_{\rho+\delta}(p)$. Since u is the solution of the initial value problem (3), (4), and $\delta = dist(E_\rho(p), K)$, the standard domain of dependence argument for initial value problems implies that u and u_t are zero on

$$\{(x,t)\ :\ x \in E_\rho(p),\ |t|<\delta\}\ . \tag{11}$$

Fix a small $\epsilon>0$, $\epsilon<\rho$, and let $q \in B_\epsilon(p) \cap H$; note $q \in E_\rho(p)$. Now u may be considered as the solution of the initial boundary value problem

$$u_{tt}-\Delta u = 0, \qquad x \in R^n \setminus D,\ t \geq \delta-\epsilon$$

$$u = f_1,\ u_t = f_2, \qquad \text{on } \{R^n \setminus D\} \times \{t=\delta-\epsilon\}\ .$$

$$u = h \text{ on } S \times [\delta-\epsilon, \infty)\ .$$

for some functions f_1 and f_2. Now f_1 and f_2 are zero on $E_\rho(p)$ (by hypothesis), h is zero on $\Gamma \times [\delta-\epsilon,\infty)$, and $d(q,x) \geq d(p,x)-d(p,q) \geq \rho-\epsilon$ for all $x \in D^c$ which are not in $E_\rho(p) \cup \Gamma$ (note Γ is the part of the boundary of $E_\rho(p)$ which lies on S). Hence from Proposition 2, $u(q,t)$ is zero for all $t \in [\delta-\epsilon, \delta-\epsilon+\rho-\epsilon)$.

Since we have already shown that u is zero on (11), we have that $u(q,t)$ is zero for all t in $[0, \rho+\delta-2\epsilon)$. Since u is odd in t we have $u(q,t)$ is zero for all t with

$|t| < \rho + \delta - 2\epsilon$, for all $q \in B_\epsilon(p) \cap H$. So, from Proposition 1, $u_t(x,0)$, and hence $f(x)$, is zero on

$$\{\ x\ :\ |x - q^*| < \rho + \delta - 2\epsilon\ \}$$

for all small $\epsilon > 0$ and a (actually any) q^* in the interior of $B_\epsilon(p) \cap H$. Hence $f(x)$ is zero on

$$\{\ x\ :\ |x - p| < \rho + \delta - 3\epsilon\ \}$$

for all $\epsilon > 0$, and hence $f(x)$ is zero on

$$\{\ x\ :\ |x - p| < \rho + \delta\ \}$$

and the theorem is proved.

3. Proof of Theorem 5

Since D is a bounded, open, connected subset of R^n, with a smooth boundary, so the complement of $\overline{D}$ will be a disjoint union of connected, open sets called components of $R^n \setminus \overline{D}$. Since D is bounded, only one of the components will be unbounded and the rest of the components will be subsets of a fixed ball in R^n. Then from the smoothness of the boundary of D and compactness, one may show that the number of components is finite, the boundaries of the components are disjoint and subsets of the boundary of D, and the boundaries are smooth.

Part 1

Let $\delta = diam(D)$ and $u = h$ on $S \times [0, \delta/2]$ (h is given to us). Since u is an odd function of t so we extend h as an odd function of t. Below $\partial_\nu u$ will represent the derivative of u on $S \times (-\infty, \infty)$ in the direction of the outward pointing normal to $S \times (-\infty, \infty)$.

Since f is supported in $\overline{D}$, we may consider u as the solution of the exterior problem

$$\begin{aligned} u_{tt} - \Delta u = 0, \qquad & \text{on } (R^n \setminus D) \times [-\delta/2,\ \delta/2] \\ u(x, t{=}0) = 0,\ u_t(x, t{=}0) = 0, \qquad & x \in R^n \setminus D \\ u = h \text{ (given)} \qquad & \text{on } S \times [-\delta/2,\ \delta/2]\ . \end{aligned}$$

This initial boundary value problem(IBVP) is well posed and so one may obtain the value of $\partial_\nu u$ on $S \times [-\delta/2,\ \delta/2]$ - this may be done numerically using finite differences (one may assume $u = 0$ for points far away from S without changing the value of $\partial_\nu u$ on $S \times [-\delta/2,\ \delta/2]$) .

Now we have u and $\partial_\nu u$ on $S \times [-\delta/2, \delta/2]$ and we show how we may recover u and u_t over the region $\overline{D} \times \{t{=} - \delta/2\}$. This is done using the Kirchhoff formula which expresses the value of a solution of the wave equation in a cylindrical (in time) domain, at a point, purely in terms of the value of the solution and its normal derivative, on the intersection of the cylinder with the forward light cone through the point - see Figure 8. This may be done only in odd space dimensions as will be seen in the formal derivation below - the derivation may be made rigorous. A rigorous derivation in the three space dimensional case may be found in [12].

Let $E_+(x,t)$ be the fundamental solution of the wave operator with support in the region $t \geq 0$ (see [14], Chapter VI). Consider a point $p \in D$. Then

$$\Box E_+(x - p, t + \delta/2) = \delta(x - p, t + \delta/2), \qquad (x,t) \in R^{n+1}\ .$$

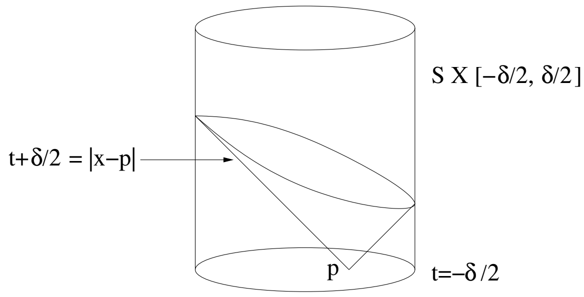


FIGURE 8. Kirchhoff Formula

Also, note that $E_+(x-p, t+\delta/2)$ is zero for $t < -\delta/2$ and is zero also on $\overline{D} \times (\delta/2, \infty)$ because the support of $E_+(x-p, t+\delta/2)$, *for odd n*, is on the cone $t+\delta/2 = |x-p|$ and $|x-p| \leq \delta$ for any $x \in \overline{D}$. Then, from Green's theorem

$$\begin{aligned}
u(p, -\delta/2) &= \int_{-\infty}^{\infty} \int_D u(x,t)\, \delta(x-p, t+\delta/2)\, dx\, dt \\
&= \int_{-\infty}^{\infty} \int_D u(x,t)\, \Box E_+(x-p, t+\delta/2)\, dx\, dt \\
&= \int_{-\infty}^{\infty} \int_D \Box u(x,t)\, E_+(x-p, t+\delta/2)\, dx\, dt \\
&\quad + \int_{-\infty}^{\infty} \int_S (\partial_\nu u(x,t)\, E_+(x-p, t+\delta/2) - u(x,t)\, \partial_\nu E_+(x-p, t+\delta/2))\, dS_x\, dt \\
&= \int_{-\delta/2}^{\delta/2} \int_S (\partial_\nu u(x,t)\, E_+(x-p, t+\delta/2) - u(x,t)\, \partial_\nu E_+(x-p, t+\delta/2))\, dS_x\, dt\ .
\end{aligned}$$

Note that the singular set of E_+ consists of the forward light cone through $(p, -\delta/2)$ and the singular directions (the Wave Front set) of E_+, away from the vertex of the cone, are the normals to the cone, and so are transverse to $S \times (-\infty, \infty)$, and hence E_+ and $\partial_\nu E_+$ have traces on $S \times (-\infty, \infty)$.

Examining the definition of E_+ in [14], Chapter VI, the last integral may be written in terms of the values of u and $\partial_\nu u$ (and their time derivatives) on $S \times [-\delta/2, \delta/2]$. Hence we now have the values of u on $D \times \{t = -\delta/2\}$ - using continuity we can determine the value on $\overline{D} \times \{t = -\delta/2\}$. A similar argument will recover the value of u_t on $\overline{D} \times \{t = -\delta/2\}$.

Knowing u and u_t on $\overline{D} \times \{t = -\delta/2\}$ and that u is the solution of the well-posed IBVP

$$u_{tt} - \Delta u = 0, \qquad \text{on } \overline{D} \times [-\delta/2, 0]$$

$$u(., t = -\delta/2) = \text{known},\ u_t(., t = -\delta/2) = \text{known}, \qquad \text{on } \overline{D}$$

$$u = h \text{ (given)} \qquad \text{on } S \times [-\delta/2, 0];$$

we may solve this numerically using finite differences and obtain the value of u_t on $\overline{D} \times \{t = 0\}$ and so obtain f.

Part II
Again δ represents the diameter of D. If u is known on $S \times [0, \delta]$ then we now give a simpler inversion scheme than the one given above. The problem is the recovery of $u_t(x, 0)$ for $x \in \overline{D}$ from the values of u on $S \times [0, \delta]$.

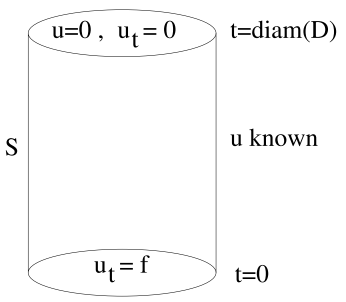


FIGURE 9. Backward IBVP

In odd space dimensions, the domain of dependence, of the value of the solution of the wave equation at a point, is the sphere of intersection of the the backward characteristic cone through that point with $t = t_{init}$. Since u is a smooth solution of the wave equation and the initial data is supported in $\overline{D}$, we have $u(x, t)$ is zero for $t \geq \delta$ and $x \in \overline{D}$. Hence u and u_t are zero on $\overline{D} \times \{t = \delta\}$. Now we may consider u as the solution of the backward IBVP

$$u_{tt} - \Delta u = 0, \qquad \text{on } \overline{D} \times [0, \delta]$$

$$u(., t{=}\delta) = 0, \;\; u_t(., t{=}\delta) = 0, \qquad \text{on } \overline{D}$$

$$u = h \qquad \text{on } S \times [0, \delta] \; .$$

This problem is well posed, so given h one may obtain $u_t(x, 0)$ for x in $\overline{D}$ and hence recover f.

4. PROOF OF THEOREM 6

We first note that (7) follows fairly quickly from (6) (but not vice versa) as shown next. Noting that (7) is symmetric it is enough to prove its norm form, namely

$$\frac{1}{2} \int_{R^3} |f(x)|^2 \, dx = \frac{1}{\rho} \int_0^\infty \int_{|p|=\rho} t \, |u_t(p, t)|^2 \, dS_p \, dt,$$

for all $f \in C_0^\infty(\overline{B_\rho(0)})$. To prove this, we take $f_1 = f_2 = f$ in (6). Then, using an integration by parts,

$$\begin{aligned}
\frac{1}{2}\int_{R^3} |f(x)|^2\,dx &= \frac{-1}{\rho}\int_0^\infty \int_{|p|=\rho} tu(p,t)\,u_{tt}(p,t)\,dS_p\,dt \\
&= \frac{1}{\rho}\int_0^\infty \int_{|p|=\rho} \{t\,u_t(p,t)\,u_t(p,t) + u(p,t)u_t(p,t)\}\,dS_p\,dt \\
&= \frac{1}{\rho}\int_0^\infty \int_{|p|=\rho} \left\{t\,|u_t(p,t)|^2 + \frac{1}{2}\frac{\partial}{\partial t}(u^2(p,t))\right\}\,dS_p\,dt \\
&= \frac{1}{\rho}\int_0^\infty \int_{|p|=\rho} t\,|u_t(p,t)|^2\,dS_p\,dt
\end{aligned}$$

where we made use of the fact that $u(p,t{=}0) = f(p) = 0$ for $|p| = \rho$ and that from Huyghen's principle (note n is odd and $n \geq 3$) $u(p,t) = 0$ for all $t > 2\rho$ and $|p| = \rho$.

To prove (6), we will first prove it in the case when $n = 3$, and then we will show (with some effort) that the case for all odd $n \geq 3$ follows from this.

4.1. **Proof of trace identity (6) when $n = 3$. Part I - An Inversion Formula** The proof of the three dimensional case is based actually on proving one of the inversion formulas in Theorem 3 directly, that is without relating it to the wave equation. Note that $\mathcal{D}$ is the identity operator when $n = 3$. We will show that for every $f \in C_0^\infty(\overline{B_\rho(0)})$,

$$f(x) = -\frac{2}{\rho}\Delta\,(\mathcal{N}^*\,t\,\mathcal{N})(f)(x), \qquad \forall x \in B_\rho(0)\ . \tag{12}$$

Below, we will make use of the following observation. Suppose $\mathcal{M}$ is an $n-1$ dimensional surface in R^n, given by $\phi(z) = 0$, with $\nabla\phi(z) \neq 0$ at every point of $\mathcal{M}$. Then

$$\int_{\mathcal{M}} h(z)\,dS_z = \int h(z)\,|\nabla\phi(z)|\,\delta(\phi(z))\,dz\ .$$

We now compute $\mathcal{N}^*(t(\mathcal{N}f)(x))$. We have

$$\begin{aligned}
(\mathcal{N}^*(t\mathcal{N}f))(x) &= \frac{1}{4\pi}\int_{|p|=\rho} \frac{1}{|x-p|}\,|x-p|\,(\mathcal{N}f)(p,|x-p|)\,dS_p \qquad (13)\\
&= \frac{1}{8\pi^2}\int_{|p|=\rho}\int_{R^3} f(y)\,\delta(|y-p|^2 - |x-p|^2)\,dy\,dS_p \\
&= \frac{1}{8\pi^2}\int_{R^3} f(y)\int_{|p|=\rho} \delta(|y-p|^2 - |x-p|^2)\,dS_p\,dy \\
&= \frac{\rho}{4\pi^2}\int_{R^3} f(y)\int_{R^3} \delta(|y-p|^2 - |x-p|^2)\,\delta(|p|^2 - \rho^2)\,dp\,dy \qquad (14)
\end{aligned}$$

The inner integral is an integral on the curve of intersection of the sphere $|p| = \rho$ with the plane of points equidistant from x and y. Define a characteristic function $\chi(x,y)$, for $x \neq y$, which is 1 if the above plane intersects the sphere $|p| = \rho$ in a circle of non-zero radius and zero otherwise.

Let Q be the orthogonal transformation which maps $y-x$ to $|y-x|e_3$ where $e_3=[0,0,1]$. Then Qx and Qy differ only in the third coordinate and in fact $Qy = Qx+|y-x|e_3$. Then, using an orthogonal change of variables, the inner integral may be rewritten as

$$\int_{R^3} \delta(|y-Q^Tp|^2-|x-Q^Tp|^2)\,\delta(|Q^Tp|^2-\rho^2)\,dp = \int_{R^3} \delta(|Qy-p|^2-|Qx-p|^2)\,\delta(|p|^2-\rho^2)\,dp\,.$$

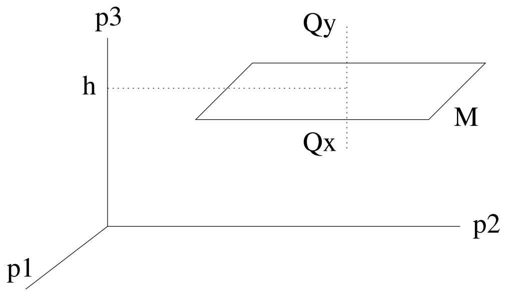


FIGURE 10. The plane $\mathcal{M}$

Let $\mathcal{M}$ be the plane consisting of points in p space which are equidistant from Qx and Qy. In fact $\mathcal{M}$ is the plane $p_3=h$ where $h=(Qx)\cdot e_3+|y-x|/2$. Further

$$\left|\nabla_p(|Qy-p|^2-|Qx-p|^2)\right| = 2\,|(p-Qy)-(p-Qx)| = 2|Qx-Qy| = 2|x-y|\,.$$

Then for x,y with $\chi(x,y)=1$, the inner integral of (14) equals

$$\frac{1}{2|x-y|}\int_{\mathcal{M}} \delta(|p|^2-\rho^2)\,dS_p\,.$$

Now $\mathcal{M}$ may be parameterized by p_1,p_2; hence the inner integral of (14) is

$$\frac{1}{2|x-y|}\int \delta(p_1^2+p_2^2+h^2-\rho^2)\,dp_1\,dp_2\,.$$

So the integral is really over the circle C centered at the origin with radius $\sqrt{\rho^2-h^2}$. Now on $p_1^2+p_2^2+h^2=\rho^2$, the magnitude squared of the gradient of $p_1^2+p_2^2+h^2-\rho^2$ is

$$4(p_1^2+p_2^2) = 4(\rho^2-h^2)\,.$$

Hence the integral equals

$$\frac{1}{2|x-y|}\int_C \frac{1}{2\sqrt{\rho^2-h^2}}\,ds = \frac{\pi}{2|x-y|}\,.$$

Hence

$$(\mathcal{N}^*(t\mathcal{N}f))(x) = \frac{\rho}{8\pi}\int \chi(x,y)\frac{f(y)}{|x-y|}\,dy\,.$$

The above calculations could be done more rigorously (i.e. without the use of δ functions) with the help of the coarea formula in [8].

Now if x and y are in the open ball $B_\rho(0)$, and $x \neq y$, then $\chi(x,y) = 1$. Hence if f is a smooth function supported in the ball $\overline{B_\rho(0)}$, then

$$(\mathcal{N}^*(t\mathcal{N}f))(x) = \frac{\rho}{8\pi}\int \frac{f(y)}{|x-y|}\,dy, \qquad \forall x \in B_\rho(0)\ . \tag{15}$$

Hence taking the Laplacian of both sides we get

$$\Delta_x(\mathcal{N}^*t\mathcal{N})(f)(x) = \frac{-4\pi\rho}{8\pi}\int f(y)\delta(x-y)\,dy = \frac{-\rho f(x)}{2}, \qquad \forall x \in B_\rho(0),$$

which implies

$$f(x) = \frac{-2}{\rho}\Delta_x(\mathcal{N}^*t\mathcal{N})(f)(x), \qquad x \in B_\rho(0) \tag{16}$$

for all smooth functions f supported in $\overline{B_\rho(0)}$.

Part II - The Identity

We now prove (6) in Theorem 6 in the $n = 3$ case. For $f_i \in C_0^\infty(\overline{B_\rho(0)})$, $i = 1,2$, let $u_i(x,t)$ be the solutions of the IVP (3), (4) with $f = f_i$. Then, from (8), $u_i(p,t) = (\mathcal{N}f)(p,t)$ for any $p \in S_\rho(0)$. Further, u_{itt} is also a solution of (3) except its initial conditions are

$$u_{itt}(x,0) = \Delta_x u_i(x,0) = 0, \qquad u_{ittt}(x,0) = \Delta_x u_{it}(x,0) = \Delta f_i(x)\ .$$

Hence $\mathcal{N}(\Delta f_i)(p,t) = u_{itt}(p,t)$ for all $p \in S_\rho(0)$ and all $t \in [0,\infty)$.

From (16) we have

$$\begin{aligned}
\frac{1}{2}\int_{R^3} f_1(x)\,f_2(x)\,dx &= \frac{-1}{\rho}\langle \Delta(\mathcal{N}^*t\mathcal{N}f_1), f_2\rangle \\
&= \frac{-1}{\rho}\langle t(\mathcal{N}f_1)(p,t), \mathcal{N}(\Delta f_2)(p,t)\rangle \\
&= \frac{-1}{\rho}\int_0^\infty \int_{|p|=\rho} t\,(\mathcal{N}f_1)(p,t)\,\mathcal{N}(\Delta f_2)(p,t)\,dS_p\,dt \\
&= \frac{-1}{\rho}\int_0^\infty \int_{|p|=\rho} t\,u_1(p,t)\,u_{2tt}(p,t)\,dS_p\,dt
\end{aligned}$$

proving (6) for the $n = 3$ case.

4.2. **Proof of trace identity (6) for all odd $n \geq 3$.** Let $\{\phi_m\}_{m=1}^\infty$ be spherical harmonics which form an orthonormal basis for $L^2(S_1(0))$ - see Chapter 4 of [29]. These are restrictions to $S_1(0)$ of some harmonic, homogeneous polynomials on R^n. If ϕ_m is the restriction of a homogeneous polynomial of degree $k(m)$ then that homogeneous harmonic polynomial is $r^{k(m)}\phi_m(\theta)$ where $r = |x|$ and $\theta = x/|x|$.

Suppose f is a smooth function on R^n supported in $\overline{B_\rho(0)}$. We have a decomposition of f of the form (convergence in L^2)

$$f(r\theta) = \sum_{m=1}^\infty f_m(r)\,r^{k(m)}\,\phi_m(\theta), \qquad r \geq 0,\ |\theta| = 1$$

with

$$r^{k(m)} f_m(r) = \int_{|\theta|=1} f(r\theta)\, \phi_m(\theta)\, d\theta\ . \tag{17}$$

From the smoothness and support of $f(x)$ we may show[1] that $f_m(r)$ is a smooth, even function on $(-\infty, \infty)$, supported in $[-\rho, \rho]$.

Below we will show that the solution, $u(x,t)$, of (3), (4), will have the form

$$u(x,t) = \sum_{m=1}^{\infty} a_m(r,t)\, r^{k(m)} \phi_m(\theta)$$

where $r = |x|$ and $\theta = x/|x|$. Then, from the orthonormality of $\{\phi_m\}_{m=1}^{\infty}$, the LHS of the trace identity (6) is

$$\begin{aligned}
\frac{1}{2}\int_0^{\infty}\int_{|\theta|=1} r^{n-1}\, f_1(r\theta)\, f_2(r\theta)\, d\theta\, dr &= \frac{1}{2}\sum_{m=1}^{\infty}\int_0^{\infty} r^{n-1} r^{2k(m)} f_{1m}(r)\, f_{2m}(r)\, dr \\
&= \frac{1}{2}\sum_{m=1}^{\infty}\int_0^{\infty} r^{\nu(m)-1} f_{1m}(r)\, f_{2m}(r)\, dr
\end{aligned}$$

where $\nu(m) = 2k(m) + n$. The RHS of (6) is

$$\begin{aligned}
RHS &= \frac{-1}{\rho}\int_0^{\infty}\int_{|p|=\rho} t\, u_1(p,t)\, u_{2tt}(p,t)\, dS_p\, dt \\
&= \frac{-1}{\rho}\sum_{m=1}^{\infty}\sum_{l=1}^{\infty}\int_0^{\infty}\int_{|p|=\rho} t\, a_{1m}(\rho,t)\, a_{2ltt}(\rho,t)\, \rho^{k(m)+k(l)}\, \phi_m(p/|p|)\, \phi_l(p/|p|)\, dS_p\, dt \\
&= -\sum_{m=1}^{\infty}\sum_{l=1}^{\infty} \rho^{k(m)+k(l)+n-2}\int_0^{\infty} t\, a_{1m}(\rho,t)\, a_{2ltt}(\rho,t)\, dt \int_{|\theta|=1}\phi_m(\theta)\, \phi_l(\theta)\, d\theta \\
&= -\sum_{m=1}^{\infty} \rho^{\nu(m)-2}\int_0^{\infty} t\, a_{1m}(\rho,t)\, a_{2mtt}(\rho,t)\, dt\ .
\end{aligned}$$

So, to prove (6), it would be enough to prove the following: if $f_i(x)$ have the form $g_i(r) r^k \phi(\theta)$ where $g_i(r)$ are smooth, even functions of r, supported in $[-\rho, \rho]$ and $\phi(x)$ is a homogeneous, harmonic polynomial on R^n of some degree k with the L^2 norm of

[1]One may show that all derivatives of the function

$$r \to \int_{|\theta|=1} f(r\theta)\, \phi_m(\theta)\, d\theta$$

up to order $k(m) - 1$ are zero at $r = 0$ because these derivatives at $r = 0$ will be sums of terms of the form

$$\int_{|\theta|=1} \theta^{\alpha}\, \phi_m(\theta)\, d\theta, \qquad |\alpha| < k(m),$$

and ϕ_m is orthogonal to all polynomials of degree less than $k(m)$, on the unit sphere (Theorem 2.1 and Corollary 2.4 of Chapter IV in [29]).

ϕ on $S_1(0)$ equal to 1, then the solution $u_i(x,t)$ has the form $a_i(r,t)r^k\phi(\theta)$ and

$$\frac{1}{2}\int_0^\infty r^{\nu-1} g_1(r)\, g_2(r)\, dr = -\,\rho^{\nu-2}\int_0^\infty t\, a_1(\rho,t)\, a_{2tt}(\rho,t)\, dt \tag{18}$$

where $\nu = n+2k$. Note that the RHS of (18) depends on ρ while the LHS does not seem to; but we assumed that the g_i were supported in $[-\rho,\rho]$.

Since $r^k\phi(\theta)$ is harmonic, if Δ_S is the Laplace-Beltrami operator on $S_1(0)$, then noting that

$$\Delta = \partial_r^2 + \frac{n-1}{r}\partial_r + \frac{1}{r^2}\Delta_S$$

one may show that

$$\Delta_S\phi = -k(k+n-2)\phi \qquad \text{on } S_1(0)\ . \tag{19}$$

When $f = g(r)r^k\phi(\theta)$, we seek a solution of (3), (4), of the form $u(x,t) = a(r,t)r^k\phi(\theta)$. Noting that

$$\begin{aligned}(ar^k)_r &= r^k a_r + kr^{k-1}a\\ (ar^k)_{rr} &= r^k a_{rr} + 2kr^{k-1}a_r + k(k-1)ar^{k-2},\end{aligned}$$

if we substitute $u = a(r,t)r^k\phi(\theta)$ in (3) and use (19), we have

$$\begin{aligned}0 &= \left((ar^k)_{tt} - (ar^k)_{rr} - \frac{n-1}{r}(ar^k)_r\right)\phi_m - \frac{ar^k}{r^2}\Delta_S\phi_m\\ &= \phi_m r^k\left(a_{tt} - a_{rr} - \frac{n+2k-1}{r}a_r\right)\ .\end{aligned}$$

Hence $a(r,t)$ must satisfy (here $\nu = n+2k$)

$$a_{tt} - a_{rr} - \frac{\nu-1}{r}a_r = 0, \qquad r\in(-\infty,\infty),\ t\geq 0 \tag{20}$$

$$a(.,t{=}0) = 0,\ a_t(.,t{=}0) = g\ . \tag{21}$$

This is an IVP for the Darboux equation which is well posed and has an explicit solution given on page 700 of [7]. Essentially, one may use a method of descent to reduce the problem to the cases $\nu = 2$ and $\nu = 3$ by noting that a_r/r also satisfies (20) and (21) except with ν replaced by $\nu+2$ and g replaced by g_r/r.

Now if n is odd then $\nu = n+2k$ is odd. So the goal is to show that for all odd $\nu = 3,5,\cdots$, and all $g_i(r)$ which are smooth, even, and supported in $[-\rho,\rho]$, we have

$$\frac{1}{2}\int_0^\infty r^{\nu-1}\, g_1(r)\, g_2(r)\, dr = r^{\nu-2}\int_0^\infty t\, a_1(r,t)\, a_{2tt}(r,t)\, dt, \qquad \forall r\geq\rho, \tag{22}$$

where $a_i(r,t)$, $i=1,2$ are the solution of (20), (21) with $g = g_i$.

Now we proved (6) for $n=3$ and hence we have proved (22) for all $\nu = 3+2k$ with $k = 0,1,2,\cdots$. Hence, we have already proved (22) for all odd ν, $\nu\geq 3$ (note (20) depends on ν and not on n directly). So we have completed the proof of (6).

Remark: Another possible approach to proving (22) without first proving (6) for the $n=3$ case is to first verify (22) for $\nu=3$ (it is easy to write the explicit solution of (20) when $\nu=3$), and then use a method of descent by observing that a_r/r also solves (20) except with ν replaced by $\nu+2$. So far, we have been unable to use the method of descent to prove (22). We were able to prove a symmetric version of this relation using the method of descent, and while the non-symmetric version easily implies the symmetric version, the validity of the converse is not known.

5. Appendix

The material is based on [8] and [10] and is included here for the reader's convenience - just Proposition 5 is new.

We give a definition of the $m-1$ dimensional Hausdorff measure on R^m. For a subset S of R^m define

$$\gamma(S) = vol(m-1)\ (diam(S)/2)^{m-1}$$

where $vol(m-1)$ is the volume of the $m-1$ dimensional unit ball. So if S were the intersection of a ball in R^m with a hyperplane then $\gamma(S)$ would be its surface area. For any positive δ, define

$$\phi_\delta(S) = \inf_{\mathcal{F}} \sum_{U \in \mathcal{F}} \gamma(U)$$

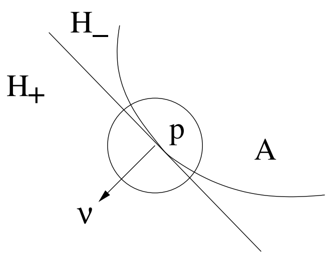
where $\mathcal{F}$ is a countable open cover of S, with each set in $\mathcal{F}$ having diameter less than δ. Now $\phi_\delta(S)$ is a decreasing function of δ (the larger the δ the greater the number of admissible open covers and hence the smaller the infimum) so we may define

$$\Phi(S) = \lim_{\delta \to 0^+} \phi_\delta(S) = \sup_{\delta > 0} \phi_\delta(S)\ .$$

It is shown in [8] that Φ is an outer measure, the σ algebra of all Borel subsets of R^m are measurable in this outer measure, and Φ is regular. Further, if a surface S is the graph of a smooth function from an open subset of R^{m-1} to R, then $\Phi(S)$ equals the usual surface area of S (Section 3.3.4 in [8]). So the Hausdorff measure generalizes the notion of surface area to Borel subsets of R^m.

Next we define the exterior normal for any subset of R^m. For a point $p \in R^m$ and a unit vector ν we define the half-planes

$$H_+(p,\nu) = \{\, x \in R^m\ :\ (x-p)\cdot\nu > 0 \,\}, \qquad H_-(p,\nu) = \{\, x \in R^m\ :\ (x-p)\cdot\nu < 0 \,\}\ .$$



Suppose A is a subset of R^m and p a point in R^m. A unit vector ν is defined to be an exterior normal to A at p if

$$\begin{aligned} \lim_{r\to 0^+} r^{-m}\ |A^c \cap H_-(p,\nu) \cap B_r(p)| &= 0, \\ \lim_{r\to 0^+} r^{-m}\ |A \cap H_+(p,\nu) \cap B_r(p)| &= 0\ . \end{aligned}$$

Here $|\ |$ is the Lebesgue measure on R^m. It is shown in [10] that if such a unit vector exists (for a given A and p) then it is unique. We denote this unit vector by $\nu(A,p)$. If no such unit vector exists then we set $\nu(A,p)=0$.

Proposition 4. *Below, a vector $x \in R^m$, will be occasionally written as $x=[x',x_m]$.*

- *If A is a subset of R^m, then $\nu(A,p)$ is zero if p is in the interior of A or $R^m \setminus A$.*
- *If $p \in \partial A$ and for some $\rho > 0$,*
$$A \cap B_\rho(p) = \{x \in B_\rho(p) \,:\, x_m > f(x')\}$$
for some C^2 function $f(x')$ of $m-1$ variables, then $\nu(A,p)$ is a positive multiple of $[\nabla f(p'),-1]$.
- *Under the conditions in the second item, for any unit vector $\theta \neq \nu(A,p)$, there is a $c>0$ so that for r small enough,*
$$\begin{aligned} r^{-m}\ |A^c \cap H_-(p,\theta) \cap B_r(p)| &> c, \\ r^{-m}\ |A \cap H_+(p,\theta) \cap B_r(p)| &> c\ . \end{aligned}$$

So, the second item asserts that $\nu(A,x)$ extends the notion of an outward pointing unit normal to arbitrary subsets of R^n. In the third item, if $\theta \neq \nu(A,p)$ then the definition of $\nu(A,p)$ implies that at least one of the limits will be non-zero - our claim is that both of them are non-zero for A with C^2 boundary.

Proof of first item

If p is an interior point of A, then for r small enough and any unit vector θ,
$$r^{-m}\ |A \cap H_+(p,\theta) \cap B_r(p)| = r^{-m}\ |H_+(p,\theta) \cap B_r(p)| = vol(m)/2 > 0,$$
and hence θ can not be $\nu(A,p)$. A similar argument works if p is an interior point of A^c.

Proof of second and third items

We will prove the result in the case $m=2$ - the general case is very similar. Here points in R^2 will be denoted by (x,y).

Without loss of generality we assume that $p=(0,0)$, that there is an $f \in C^2(R)$ with $f(0)=0$, $f'(0)=0$, and that
$$A = \{\,(x,y) \,:\, y > f(x)\,\}\ .$$
Hence we have the representation $f(x)=x^2 g(x)$ for some continuous function $g(x)$.

Since $B_r(0)$ contains the rectangle $[-r/2,r/2]\times[-r/2,r/2]$ and is contained in the rectangle $[-r,r]\times[-r,r]$, WLOG we may assume that $B_r(p)$ is the rectangle $[-r,r]\times[-r,r]$. Further, we may take r small enough so that $|f(x)|<r$ for $|x|<r$.

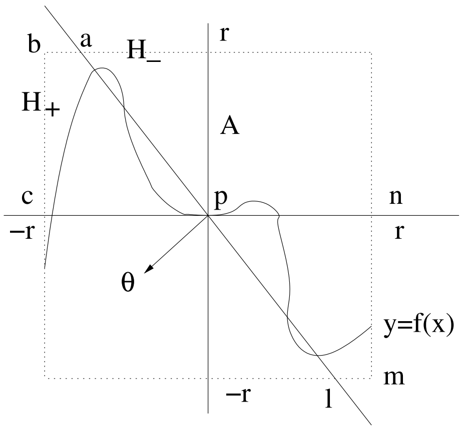
We first show that $\nu(A,p)=e_2=(0,-1)$. Now $H_+(e_2,p)$ is the lower half plane and $H_-(e_2,p)$ is the upper half plane. Then
$$\begin{aligned} A \cap B_r(p) \cap H_+(e_2,p) &= \{\,(x,y)\,;\, -r<x<r,\ \min(f(x),0) \leq y \leq 0\,\}, \\ A^c \cap B_r(p) \cap H_-(e_2,p) &= \{\,(x,y)\,;\, -r<x<r,\ 0 \leq y \leq \max(f(x),0)\,\},\ . \end{aligned}$$

Hence

$$
\begin{aligned}
r^{-2}\left|A \cap B_r(p) \cap H_+(e_2,p)\right| &\leq r^{-2}\int_{-r}^{r} |f(x)|\,dx \leq Cr^{-2}\int_{-r}^{r} x^2\,dx = \frac{2Cr}{3}\\
r^{-2}\left|A^c \cap B_r(p) \cap H_-(e_2,p)\right| &\leq r^{-2}\int_{-r}^{r} |f(x)|\,dx \leq Cr^{-2}\int_{-r}^{r} x^2\,dx = \frac{2Cr}{3},
\end{aligned}
$$

which proves the second item.

We now give a proof of the third item. We will give a proof when $\theta = (\theta_1, \theta_2)$ with $\theta_1 < 0$ and $\theta_2 < 0$. The other cases are similar. Below we will talk of the quadrilaterals (or triangles) $Quad(pabc)$ and $Quad(plmn)$ which represent the intersections of $H_+(p,\theta)$ and $H_-(p,\theta)$ with the second and fourth quadrants.



We observe that

$$
\begin{aligned}
&A \cap H_+(p,\theta) \cap B_r(p) \supset Quad(pabc) \setminus A^c\\
&\quad = \; Quad(pabc) \setminus \{\, (x,y) \,:\, -r < x < 0,\; 0 \leq y \leq \max(f(x),0) \,\},\\
&A^c \cap H_-(p,\theta) \cap B_r(p) \supset Quad(plmn) \setminus A\\
&\quad = \; Quad(plmn) \setminus \{\, (x,y) \,:\, 0 < x < r,\; \min(f(x),0) \leq y \leq 0 \,\}\,.
\end{aligned}
$$

Hence

$$
\begin{aligned}
r^{-2}\left|A \cap H_+(p,\theta) \cap B_r(p)\right| &\geq r^{-2}Area(pabc) - r^{-2}\int_{-r}^{0} |f(x)|\,dx\\
r^{-2}\left|A^c \cap H_-(p,\theta) \cap B_r(p)\right| &\geq r^{-2}Area(plmn) - r^{-2}\int_{0}^{r} |f(x)|\,dx\,.
\end{aligned}
$$

Now $r^{-2}Area(pabc) = r^{-2}Area(plmn) = C$ for some constant $C > 0$ independent of r, and

$$
r^{-2}\int_{-r}^{r} |f(x)|\,dx \leq C_1 r^{-2}\int_{-r}^{r} x^2\,dx = \frac{2C_1 r}{3}\,.
$$

Hence the result follows. **QED**

For subsets A and B of R^m, let $p \in \partial(A \cap B)$. We now wish to relate $\nu(A \cap B, p)$ to $\nu(A,p)$ and $\nu(B,p)$. If $p \in \partial(A \cap B)$ then $p \in \partial A \cup \partial B$ and if p is not a boundary point of B then it is an interior point of B and hence $B_r(p) \cap (A \cap B) = B_r(p) \cap A$ for

r small enough. Hence for boundary points p of $A \cap B$, with $p \notin \partial A \cap \partial B$, we have $\nu(A \cap B, p) = \nu(A, p)$ if $p \in \partial A$ and $\nu(A \cap B, p) = \nu(B, p)$ if $p \in \partial B$. So it remains to determine $\nu(A \cap B, p)$ when $p \in \partial A \cap \partial B$.

Proposition 5. *Suppose A and B are subsets of R^m, p a boundary point of $A \cap B$, and $p \in \partial A \cap \partial B$. Suppose, for some $\rho > 0$,*

$$A \cap B_\rho(p) = \{x \in B_\rho(p) \,:\, x_m > f(x')\}$$

for some C^2 function $f(x')$ of $m-1$ variables then either $\nu(A \cap B, p) = \nu(A, p)$ or $\nu(A \cap B, p) = 0$.

Proof

Let θ be a unit vector, $\theta \neq \nu(A, p)$. Then from Proposition 4, there is a $c > 0$, so that for small enough r

$$r^{-m} \ |A^c \cap H_-(p, \theta) \cap B_r(p)| \ > \ c \ .$$

Hence, for small enough r

$$r^{-m} \ |(A \cap B)^c \cap H_-(p, \theta) \cap B_r(p)| \ > \ c \ .$$

So θ can not be the normal to $A \cap B$ at p. **QED**

Now we state the Gauss-Green theorem as stated in [10].

Proposition 6 (Gauss-Green Theorem). *Let A be a bounded measurable subset of R^m with $\Phi(\partial A) < \infty$, and $f \in C^1(R^m)$. Then*

$$\int_A \frac{\partial f}{\partial x_j} \, dx = \int_{R^m} f(x) \, \nu_j(A, x) \, d\Phi \qquad\qquad j = 1, 2, \cdots, m$$

Here $\nu_j(A, x)$ is the jth component of $\nu(A, x)$.

References

[1] M L Agranovsky and E T Quinto. Injectivity Sets for the Radon Transform over Circles and Complete Systems of Radial Functions, Journal of Functional Analysis, 139, 383-414, (1996).
[2] M L Agranovsky and E T Quinto. Geometry of stationary sets for the wave equation in R^n; the case of finitely supported initial data, Duke Math. J. 107 (2001), no. 1, 57–84.
[3] M L Agranovsky, C Berenstein, and P Kuchment. Approximation by spherical means in L^p spaces. J. Geom. Analysis, Vol. 6, (1996), 365-383.
[4] Lars-Erik Andersson. On the determination of a function from spherical averages. SIAM J Math Anal 19 (1988), no 1, 214-232.
[5] A L Bukhgeim and V B Kardakov. Solution of an inverse problem for an elastic wave equation by the method of spherical means, Siberian Math. J. 19 (1978), no. 4, 528–535 (1979).
[6] A M Cormack and E T Quinto. A Radon transform on spheres through the origin in R^n and applications to the Darboux equation, Trans. AMS, vol 260, 1980, pp 575–581.
[7] R Courant and D Hilbert. Methods of Mathematical Physics, Volume II, John Wiley, 1962.
[8] L C Evans and R F Gariepy. Measure Theory and Fine Properties of Functions, CRC Press, 1992.
[9] J A Fawcett. Inversion of N-dimensional spherical averages, SIAM J Appl. Math. 45, 336-41, (1985).
[10] H Federer. A note on the Gauss-Green theorem. Proc. Amer. Math. Soc. 9, 447–451 (1958).
[11] D Finch and Rakesh. Trace Regularity Of Solutions Of the Wave Equation, in preparation (2002).

[12] F G Friedlander. The wave equation on a curved space-time, Cambridge University Press, 1975.
[13] A B Goncharov. Differential equations and integral geometry, Adv. Math., 131, 279-343 (1997).
[14] L Hormander. The Analysis of Linear Partial Differential Operators I, Springer-Verlag, 1983.
[15] F John. Plane Waves and Spherical Means, Wiley, 1955.
[16] F John. Partial Differential Equations, 4th edition, Springer Verlag, (1982).
[17] W Joines, Y Zhang, C Li, R Jirtle. Medical Physics 1994; 21(4):547-550.
[18] R A Kruger, K D Miller, H E Reynolds, W L Kiser Jr, D R Reinecke, G A Kruger. Contrast enhancement of breast cancer in vivo using thermoacoustic CT at 434 MHz. Radiology 2000; 216: 279-283
[19] R A Kruger, K K Kopecky, A M Aisen, D R Reinecke, G A Kruger, W L Kiser Jr. Thermoacoustic CT with radio waves: a medical imaging paradigm. Radiology 1999; 211: 275-278
[20] R A Kruger, D R Reinecke, G A Kruger. Thermoacoustic computed tomography. Medical Physics 1999; 26(9): 1832-1837
[21] A K Louis and E T Quinto. Local tomographic methods in SONAR, in "Surveys on solution methods for inverse problems" edited by D Colton, H W Engl, A K Louis, J R McLaughlin, and W Rundell, Springer-Verlag, Vienna, 147-154 (2000).
[22] M M Lavrentiev, V G Romanov, V G Vasiliev. Multidimensional Inverse Problems For Differential equations, Lecture Notes in Mathematics, Vol 167, Springer-Verlag (1970).
[23] M M Lavrentiev, V G Romanov, S P Shishatskii. Ill-posed Problems of Mathematical Physics and Analysis, Translation of Mathematical Monographs, Vol 64, American Mathematical Society (1986).
[24] J M Lee. Riemannian Manifolds, Springer, 1997.
[25] F Natterer. The mathematics of computerized tomography, Teubner 1986.
[26] S J Norton. Reconstruction of a two-dimensional reflecting medium over a circular domain: Exact Solution, J. Acoust. Soc. Am. vol 67(4), (1980), 1266-1273.
[27] S J Norton and M Linzer. Ultrasonic reflectivity imaging in three dimensions: exact inverse scattering solutions for plane, cylindrical, and spherical apertures, IEEE Transactions on Biomedical Engineering, Vol. BME-28, pp. 200-202, 1981.
[28] A G Ramm. Injectivity of the spherical mean operator, 7 pages, preprint, 2002.
[29] E M Stein, G Weiss. Introduction to Fourier Analysis on Euclidean Spaces, Princeton University Press, Princeton NJ (1971).
[30] D Tataru. Unique continuation for partial differential operators with partially analytic coefficients, J. Math. Pures Appl. **78(5)**, 505–521 (1999).
[31] V V Volchkov. Injectivity sets for the Radon Transform over a sphere, Izvestiya Math., 63 (3), 481–493, (1999).
[32] M Xu, L V Wang. Time-Domain Reconstruction for Thermoacoustic Tomography in a Spherical Geometry, IEEE Transactions in Medical Imaging, **21** no. 7, 814-822, (2002).
[33] Y Xu, D Feng, L V Wang. Exact Frequency-Domain Reconstruction for Thermoacoustic Tomography - I: Planar Geometry, IEEE Transactions in Medical Imaging, **21** no. 7, 823-828, (2002).
[34] Y Xu, M Xu, L V Wang. Exact Frequency-Domain Reconstruction for Thermoacoustic Tomography - II: Cylindrical Geometry, IEEE Transactions in Medical Imaging, **21** no. 7, 829-833, (2002).
[35] A E Yagle. Inversion of spherical means using geometric inversion and Radon transforms, Inverse problems 8, 949-964 (1992).

David Finch, Department of Mathematics, Oregon State University, Corvallis, OR 97331-4605

E-mail address: finch@math.orst.edu

Sarah Patch, G E Medical Systems, Mail Stop W-875, 3200 N Grandview Boulevard, Waukesha, WI 53188

E-mail address: sarah.patch@med.ge.com

Rakesh, Department of Mathematics, University of Delaware, Newark, DE 19716

E-mail address: rakesh@math.udel.edu