PARTIAL UNIQUENESS AND OBSTRUCTION TO UNIQUENESS IN INVERSE PROBLEMS FOR ANISOTROPIC ELASTIC MEDIA

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ABSTRACT. We consider the dynamic inverse problem of uniquely determining the material parameters of bounded, three-dimensional elastic objects from displacement and traction measurements made at their surfaces. These surface measurements are modeled by the Dirichlet-to-Neumann map on a finite time interval. We first construct an obstruction to uniqueness for the inverse problem by transforming the medium via diffeomorphisms that fix the boundary to first order. We study the equivalence classes of elastic media under the action of pullback via these diffeomorphisms. This action leaves the Dirichlet-to-Neumann map invariant. We then show that any uniqueness result for a particular type of elastic media extends to (partial) uniqueness on the entire class. For example, we establish that uniqueness for isotropic elastic media implies the unique determination by surface measurements of three parameters of some anisotropic elastic media.

1. INTRODUCTION

In the inverse problem for the scalar wave equation associated with a Riemannian metric $\mathbf{G}$,

$$(\partial_t^2 \ - \ \Delta_{\mathbf{G}})u = 0, \tag{1}$$

there is an "obstruction to uniqueness" in the dependence of the metric $\mathbf{G}$ on surface measurements. In fact, for surface measurements in the form of the Dirichlet-to-Neumann map $\Lambda_{\mathbf{G}}$ we have

$$\Lambda_{\psi^*\mathbf{G}} \ = \ \Lambda_{\mathbf{G}} \tag{2}$$

2000 *Mathematics Subject Classification.* Primary: 65M32, 74B05, 70G45; Secondary: 74B20, 74A20, 15A72.

Key words and phrases. Elastodynamics, inverse problems, parameter identification, uniqueness, covariance, anisotropic media, elasticity tensor, harmonic decomposition, Cartan decomposition, Dirichlet-to-Neumann map.

The first author was partially supported by an MSRI Postdoctoral Fellowship. Research at MSRI is supported in part by NSF grant DMS-9850361. This work was conducted while the author was a Gibbs Instructor at Yale University.

The second author was partially supported by an MSRI Postdoctoral Fellowship and by NSF grant DMS-9801664 (9996350).

for any sufficiently regular diffeomorphism $\psi : \bar{\Omega} \to \bar{\Omega}$ with $\psi(x) = x$ for $x \in \partial\Omega$. Here $\Omega \subset \mathbb{R}^3$ is an open region with smooth boundary $\partial\Omega$, and $\Delta_{\mathbf{G}}$ is the Laplace-Beltrami operator associated with the metric $\mathbf{G}$ on Ω; in local coordinates,

$$\Delta_{\mathbf{G}} u \;=\; \sum_{i,j=1}^{n} \frac{1}{\sqrt{\det \mathbf{G}}} \, \frac{\partial}{\partial x^i} \Big(\sqrt{\det \mathbf{G}} \; G^{ij} \, \frac{\partial u}{\partial x^j} \Big), \tag{3}$$

where $\det \mathbf{G}$ denotes the determinant, and G^{ij} denotes the inverse of the metric tensor G_{ij}. In other words, the best uniqueness result possible in the inverse problem for the operator $\partial_t^2 - \Delta_{\mathbf{G}}$ is the unique determination of $\mathbf{G}$ by the Dirichlet-to-Neumann map, up to the pullback by a diffeomorphism ψ that fixes the boundary. In fact, this result has been obtained for the scalar wave operator, under certain conditions, in [Sy-U] and [L-S-U], for example.

The scalar wave equation (1) is the canonical model for wave propagation in anisotropic media, and so it is natural to ask whether the uniqueness result for $\partial_t^2 - \Delta_{\mathbf{G}}$ extends to more general models, for example, the system for elastodynamics. The uniqueness question for general anisotropic elastic media, though, is significantly more difficult than that for the scalar wave equation, due mainly to greater complexity in the description of the rays, paths along which disturbances propagate through the medium. We show here, however, that several aspects of the uniqueness question may be solved in the systems case without reference to rays.

In fact, given the particular type of obstruction (2) in the scalar case, we make our first aim the description of equivalence classes of elastic media that are invariant under the Dirichlet-to-Neumann map and are defined in terms of an action by diffeomorphisms of Ω that fix the boundary. We give a pointwise description of parts of the orbits under this action, and a complete pointwise characterization in the isotropic case. (At any point of Ω the orbit of isotropic elastic media consists of isotropic, some transverse isotropic, and some orthotropic elastic media.)

Global constraints might place additional limits on the types of anisotropy within the orbit of any particular elastic media. The approach taken here does not extend directly to the global setting, as it depends on diagonalization of the symmetric part (i.e. the *stretch factor* of the polar decomposition) of the differential $D\psi$.

It is interesting to note that composite elastic media, elastic media for which the type of anisotropy can vary from point to point, may lie on the orbits of non-composite elastic media under this action.

We then observe that uniqueness results for classes of elastic media extend to the orbits under this action. We conclude that Rachele's uniqueness results [R I], [R II], [R III], [R IV] for isotropic elastodynamics extend to certain anisotropic elastic media.

We mention other results on obstruction to uniqueness in inverse problems using Dirichlet-to-Neumann-type data. In a 2-dimensional static problem for orthotropic elastic media G. Nakamura and K. Tanuma [N-T '96] show that the Dirichlet-to-Neumann map does not uniquely determine the elastic parameters at the boundary. Also, M. Belishev [B] describes an obstruction to uniqueness in the dependence of the Dirichlet-to-Neumann map on lower-order parameters of an (isotropic) wave equation. In addition, Romanov [R-K] proves nonuniqueness in an inverse problem for anisotropic Maxwell's equations in the case that the permittivity ϵ and permeability μ are diagonal matrices.

The only uniqueness result known to the authors for the fully three-dimensional inverse parameter identification problem for nonhomogeneous anisotropic elastic media using Dirichlet-to-Neumann-type data is that of Nakamura, Tanuma, and Uhlmann [N-T-U] who show that, in the case of transverse isotropic elasticity, the material properties of bounded elastic objects are uniquely determined by static measurements made at the surface. Nakamura, Tanuma, and Uhlmann use a layer-stripping approach (applicable to static problems).

To extend the non-uniqueness result (2) for the scalar wave equation (1) to elastic media we first describe (2) in terms of the covariance of the wave operator, i.e.,

$$\psi^*([\partial_t^2 - \Delta_{\mathbf{G}}]u) \;\; = \;\; [\partial_t^2 \; - \; \Delta_{\psi^*\mathbf{G}}](\psi^* u),$$

where ψ^* is the pullback under the diffeomorphism ψ. (By covariance we mean an intrinsic tensorial representation independent of any local coordinate system.) We then view the elastic body as a Riemannian manifold endowed with a smooth Riemannian metric $\mathbf{G}$ which we may vary as we vary the other material parameters. We write the differential operator for hyperelastic media in a covariant form (see Marsden and Hughes [M-H]) so that a solution U of the elastodynamics system (with respect to a metric $\mathbf{G}$ and material parameters $\mathcal{O}$) transforms to a solution $\psi^*U = U \circ \psi$ with respect to $\psi^*\mathbf{G}$ and transformed parameters $\psi^*\mathcal{O}$ (cf. Theorem 1). We conclude that the elastic objects $(\Omega, \mathbf{G}, \mathcal{O})$ and $(\Omega, \psi^*\mathbf{G}, \psi^*\mathcal{O})$ cannot be distinguished by surface measurements given by the associated Dirichlet-to-Neumann maps, i.e. $\Lambda_{\mathbf{G},\mathcal{O}} = \Lambda_{\psi^*\mathbf{G},\psi^*\mathcal{O}}$, if ψ preserves the boundary. In fact, in the classical linear case this obstruction to uniqueness may be stated with respect to the Euclidean metric:

$$\Lambda_{\mathbf{e},\sqrt{\det \mathbf{G}}\ \mathcal{O}} f = \Lambda_{\mathbf{e},\det(D\psi)\sqrt{\det \mathbf{G}}\ \psi^*\mathcal{O}} f.$$

To write the elasticity equations in covariant form in a way that respects the underlying physics, we turn to the balance laws (conservation of mass, balance of momentum, balance of moment of momentum, and balance of energy) that describe the behavior of the elastic medium. In the case of non-linear elastodynamics Marsden and Hughes [M-H], Theorem 4.13, show that the balance laws, together with the hyperelasticity condition (that is, the existence of an internal energy E), are equivalent to the covariance of the internal energy. The balance laws (in particular, the equations of motion) may be written in either the so-called *Eulerian* (or *spatial*) *coordinates* x on the ambient space $\mathcal{S}$ (here $\mathcal{S} = \mathbb{R}^3$), or in *Lagrangian* (or *material* or *referential*) coordinates X on Ω, at least for sufficiently regular motions. (See Section 3.) Here we use Lagrangian coordinates in order to describe changes in the material properties of the body in an intrinsic way, while avoiding the construction of global transformations of the ambient space.

In Lagrangian coordinates the energy E is covariant, in the case that it depends functionally on just the Cauchy-Green deformation tensor $\mathbf{C}$, only if the elastic medium is isotropic. To include anisotropic media in the analysis, therefore, we view E as a function of $\mathbf{C}$ and structural tensors $\zeta_1, \ldots, \zeta_m$ which express the specific anisotropy of the body. (See the proof of Theorem 1 in Section 4.)

For classical linear elasticity the (Lagrangian) covariant form for the equations of motion derived from the balance laws is given by [M-H]

$$P_{\mathbf{G},\mathsf{C},\rho}\mathbf{U} \;=\; \rho\frac{\partial^2 \mathbf{U}}{\partial t^2} \;-\; \mathrm{DIV}_{\mathbf{G}}(\mathsf{A}\cdot\nabla\mathbf{U}) \;=\; 0, \tag{4}$$

where $\rho(X)$ is the mass density in the reference configuration, A depends on the elasticity tensor C via

$$\mathsf{A}^{aA}{}_{b}{}^{B} = 2\mathsf{C}^{EAFB}(D\iota)^{a}{}_{E}(D\iota)^{c}{}_{F}\delta_{bc}, \tag{5}$$

ι is an embedding of Ω in $\mathbb{R}^3$, $\mathbf{U}$ is the displacement vector from a material point X in the reference configuration to its new location after time t, $\mathrm{DIV}_{\mathbf{G}}(\mathsf{A}\cdot\nabla\mathbf{U})$ is the divergence on $(\Omega, \mathbf{G})$ of the two-point tensor $\mathsf{A}\cdot\nabla\mathbf{U}$, and ∇ is the covariant derivative induced by $\mathbf{G}$. We show in the proof of Theorem 1 that the equations of motion in local coordinates are given by

$$\rho\frac{\partial^2 U^a}{\partial t^2} \;-\; \frac{1}{\sqrt{\det\mathbf{G}}}\,\frac{\partial}{\partial X^A}\left(\sqrt{\det\,\mathbf{G}}\;\;\mathsf{A}^{aA}{}_{b}{}^{B}\frac{\partial U^b}{\partial X^B}\right) \;=\; 0, \tag{6}$$

where $\det\,\mathbf{G}$ denotes the determinant of the metric tensor G_{ij}. Here an uppercase (i.e. capitalized) index denotes the leg of a tensor in the material setting, while a lowercase index refers to the spatial setting. (See Section 3 for definitions and notation.) No lower-order-terms associated to the Christoffel symbols of $\mathbf{G}$ appear, since ∇ acts on the two-point vector $\mathbf{U}$ as if it were a scalar. We emphasize that the equations under a change of (Lagrangian) coordinates ψ have the same form but with each of $\mathbf{G}, \mathsf{C}$, and ρ replaced by its pullback.

To illustrate that not all covariant forms of a differential equation respect the underlying physics, we turn to the following example. The Navier-Stokes equations of fluid mechanics

$$\begin{cases} \partial_t u - \Delta u + u \cdot \nabla u - \nabla p = 0, \\ \operatorname{div} u = 0, \end{cases} \tag{7}$$

are in covariant form if Δ is interpreted as the Laplace-Beltrami operator and ∇ is the covariant derivative. In (7) u is a vector field describing the velocity of a fluid particle, and p is a scalar field representing the pressure.

However the correct form of the equation as derived from conservation of momentum and mass is [E-M]:

$$\begin{cases} \partial_t u - \operatorname{div} S + u \cdot \nabla u - \nabla p = 0, \\ \operatorname{div} u = 0. \end{cases} \tag{8}$$

Above, $S = 1/2(\nabla u + \nabla^T u)$ is the deformation tensor and, for divergence-free vector fields u on a Riemannian manifold, $\operatorname{div} S = \Delta\, u + 2\,\mathrm{Ric}\; u$, where Ric is the Ricci tensor. In fact, one expects curvature to enter into the dynamics whenever viscous stresses are present.

The paper is organized as follows. In Section 2 we state the main results. In Section 3 we present notation and definitions, and write the equations of motion in covariant form for the non-linear, linearized, and classical linear cases. Obstruction to uniqueness (Theorem 1) is proved in Section 4. Although we focus in this paper on results for classical linear elastodynamics, our approach is based on a linearization of the non-linear equations of motion, and so we prove Theorem 1 for the non-linear and linearized cases as well. In Section 5 we prove Theorem 11 on the pointwise characterization of orbits in some cases.

2. Main Results

We say that a three-dimensional linearly elastic object (represented by $\Omega \subseteq \mathbb{R}^3$) is *admissible* if the object is bounded, with smooth boundary; has smooth, positive density ρ; and has smooth elasticity tensor C which has the symmetry properties

$$\mathsf{C}^{ABCD} = \mathsf{C}^{CDAB} = \mathsf{C}^{BACD} = \mathsf{C}^{ABDC} \tag{9}$$

and is (uniformly) strongly elliptic on Ω, i.e. there is a constant $c > 0$ so that for any $X \in \Omega$ $\mathsf{C}^{ABCD}(X) V^A V^B W^C W^D \geq c|\mathbf{V}|^2|\mathbf{W}|^2$ for all vectors $\mathbf{V}, \mathbf{W} \in T^*_X\Omega$. (The symmetry condition $\mathsf{C}^{ABCD} = \mathsf{C}^{CDAB}$ is equivalent to hyperelasticity in the linear case.) Throughout the paper we follow the convention of summing over repeated indices.

In the case of non-linear elasticity we also assume that there exist unique solutions ϕ_t of the initial-boundary-value problem (30) for regular-enough Dirichlet data f. Some partial results for well-posedness of the non-linear initial-boundary-value problem (30) are discussed, for example, in [M-H], p. 401, 406.

We model the behavior of any admissible nonhomogeneous elastic object $\Omega \subseteq \mathbb{R}^3$ by solutions of the equations of motion, e.g. $P_{\mathbf{G},\mathcal{O}}\mathbf{U} = 0$ for classical linear elastodynamics, where $\mathcal{O}$ (given by $(\mathbf{C}, \rho)$ in this case) represents the material properties of the object.

The surface measurements used to determine the material parameters $\mathcal{O}$ are given by the correspondence between forces exerted on the surface and the resulting surface displacements, taken over a finite period of time. These surface measurements are represented by the dynamic Dirichlet-to-Neumann map $\Lambda_{\mathbf{G},\mathcal{O}}$. In the case of classical linear elasticity (4) we define the Dirichlet-to-Neumann map by

(10)
$$(\Lambda_{\mathbf{G},\mathbf{C},\rho} f)^a \;=\; \langle\, \mathbf{A}\cdot\nabla\mathbf{U},\ \boldsymbol{\nu}_{\mathbf{G}}\,\rangle^a\, dS_{\mathbf{G}} \;=\; \mathbf{A}^{aA}{}_b{}^B\ (\nabla\mathbf{U})^b{}_B\ \mathbf{G}_{AC}\ \boldsymbol{\nu}_{\mathbf{G}}^C\ dS_{\mathbf{G}},$$

where $dS_{\mathbf{G}}$ is the area form on $\partial\Omega$ induced by $\mathbf{G}$, $\boldsymbol{\nu}_{\mathbf{G}}$ is the unit outer normal vector to the boundary $\partial\Omega$ with respect to $\mathbf{G}$, $\mathbf{A}$ is defined in (5), and $\mathbf{U}$ solves the initial-boundary-value problem

$$\begin{aligned}
&\text{(11a)} && P_{\mathbf{G},\mathbf{C},\rho}\mathbf{U} &&= \rho\,\frac{\partial^2\mathbf{U}}{\partial t^2} - \mathrm{DIV}_{\mathbf{G}}(\mathbf{A}\cdot\nabla\mathbf{U}) = 0 && \text{on } \Omega,\\
&\text{(11b)} && \mathbf{U}(X,t) &&= f(X,t) && \text{for } X \in \partial\Omega,\\
&\text{(11c)} && \mathbf{U}(X,t) &&= 0 && \text{for } t = 0,\\
&\text{(11d)} && \frac{\partial\mathbf{U}}{\partial t}(X,t) &&= 0 && \text{for } t = 0,
\end{aligned}$$

with Dirichlet boundary data f. The Dirichlet-to-Neumann map is defined by (33) and (38) in the cases of non-linear and linearized elastodynamics, respectively.

Given any diffeomorphism ψ of $\bar{\Omega}$ that fixes the boundary to first order, i.e. $\psi \in \Psi$, where

$$\Psi \;=\; \{\ \psi\ :\ \bar{\Omega} \to \bar{\Omega}\ \mid\ \psi|_{\partial\Omega} = id,\quad D\psi|_{\partial\Omega} = I\ \},$$

we define $\psi^*\mathcal{O}$ to be the transformed material parameters under ψ. That is, in the classical linear case $\psi^*\mathcal{O} = (\psi^*\mathbf{C}, \psi^*\rho)$. The transformed parameters $(\psi^*\mathbf{G}, \psi^*\mathbf{C}, \psi^*\rho)$ give rise to an admissible hyperelastic object.

In the proof of Theorem 1 (in Section 4) we first show that the equations of motion are covariant in the non-linear case, and we then linearize following [M-H]. Consequently, if $\mathbf{U}$ is a solution of the initial-boundary-value problem (11) for linear elastodynamics with Dirichlet boundary data f, metric $\mathbf{G}$, and material parameters $\mathcal{O}$, then $\psi^*\mathbf{U} = U \circ \psi$ is a solution of (11) with metric $\psi^*\mathbf{G}$ and material parameters $\psi^*\mathcal{O}$. We conclude, given the definition (10) of the Dirichlet-to-Neumann map and the fact that ψ is the identity to first order at the boundary, that the Dirichlet-to-Neumann maps for $(\mathbf{G}, \mathcal{O})$ and $(\psi^*\mathbf{G}, \psi^*\mathcal{O})$ agree. That is, the two elastic objects $(\Omega, \mathbf{G}, \mathcal{O})$, $(\Omega, \psi^*\mathbf{G}, \psi^*\mathcal{O})$ cannot be distinguished by boundary measurements.

Theorem 1 (Obstruction to uniqueness for hyperelastic media)**.** *Consider any admissible (linear, linearized, or non-linear) elastic body* $(\Omega, \mathbf{G}, \mathcal{O})$ *with material parameters given by* $\mathcal{O}$. *Then for any* $\psi \in \Psi$ *and the transformed parameters* $\psi^*\mathcal{O}$ *under* ψ

$$\Lambda_{\mathbf{G},\mathcal{O}} \;=\; \Lambda_{\psi^*\mathbf{G},\psi^*\mathcal{O}}.$$

In the classical linear case the obstruction may be stated with respect to the Euclidean metric:

Corollary 2 (Obstruction for linear elastodynamics with respect to the Euclidean metric)**.** *Consider any admissible linearly elastic body* $(\Omega, \mathbf{G}, \mathbf{C}, \rho)$. *Then*

$$(12)\qquad \begin{aligned} (\Lambda_{\mathbf{G},\mathbf{C},\rho} f\,)^a \;&=\; \mathbf{A}^{aA}{}_b{}^B\;(\nabla \mathbf{U})^b{}_B\;\delta_{AC}\;\boldsymbol{\nu}^C\;\sqrt{\det\mathbf{G}}\;\;dS \\ &=\; (\Lambda_{\mathbf{e},\sqrt{\det\mathbf{G}}\;(\mathbf{C},\rho)}\, f\,)^a, \end{aligned}$$

where dS *is the area form on* $\partial\Omega$ *induced by the Euclidean metric* $\mathbf{e}$, *and* $\boldsymbol{\nu}$ *is the unit outer normal vector to the boundary* $\partial\Omega$ *with respect to* $\mathbf{e}$.

In fact, for any $\psi \in \Psi$

$$(13)\qquad \Lambda_{\mathbf{e},\sqrt{\det\mathbf{G}}\,(\mathbf{C},\rho)} \;=\; \Lambda_{\mathbf{e},(\det D\psi)\sqrt{\det\mathbf{G}}\;\;(\psi^*\mathbf{C},\psi^*\rho)}.$$

Proof. The first equality in (12) follows from an application of the divergence theorem (e.g., [Ta I], p. 127). In fact, on the one hand,

$$dS_{\mathbf{G}} = \sqrt{\det\mathbf{G}} < \boldsymbol{\nu}, \boldsymbol{\nu} >_{\mathbf{G}}^{1/2}\;dS,$$

where $< \boldsymbol{\nu}, \boldsymbol{\nu} >_{\mathbf{G}} = G_{AB}\nu^A\nu^B$. On the other, the unit (co)normal to $\partial\Omega$ w.r.t. $\mathbf{G}$ is related to $\boldsymbol{\nu}$ by

$$(\nu_{\mathbf{G}})_A = < \boldsymbol{\nu}, \boldsymbol{\nu} >_{\mathbf{G}}^{-1/2}\;\nu_A.$$

The second equality is a consequence of (6), which may be viewed as a global representation of the equations of motion (4) in Euclidean coordinates. Theorem 1 then directly implies (13). □

2.1. **Partial Uniqueness for Anisotropic Linearly Elastic Media.**

We consider the orbits $\mathfrak{O}_{(\mathbf{G},\mathbf{C},\rho)} = \{\,(\psi^*\mathbf{G}, \psi^*\mathbf{C}, \psi^*\rho) \mid \psi \in \Psi\,\}$ of parameters $(\mathbf{G}, \mathbf{C}, \rho)$ of linearly elastic media under the action of Ψ. Below we prove uniqueness of some parameters for anisotropic, linearly elastic media in the orbit of isotropic elastic media and in the orbit of isotropic elastic media with residual stress.

The elasticity tensor for isotropic elastic media with respect to the Euclidean metric $\mathbf{e}$ (which we denote by $\mathbf{C}_{iso}$) has the following components:

$$(14)\qquad \mathsf{C}_{iso}^{ABCD} \;=\; \lambda(X)\delta^{AB}\delta^{CD} \;+\; \mu(X)\left[\delta^{AC}\delta^{BD} \;+\; \delta^{AD}\delta^{BC}\right]$$

with λ, μ the *Lamé parameters.* In this case we denote the Dirichlet-to-Neumann map by $\Lambda_{\mathbf{e},\lambda,\mu,\rho}$. Similarly, $\widehat{\mathbf{C}}_{iso}$ denotes the isotropic elasticity tensor with Lamé parameters $\widehat{\lambda}$ and $\widehat{\mu}$. In general, $\mathbf{C}$ depends on the metric $\mathbf{G}$.

We first recall Rachele's uniqueness result for linear, isotropic elastic media in Euclidean space.

Theorem 3 (Rachele [R I], [R II], [R III], Uniqueness for isotropic elastodynamics in Euclidean space)**.** *Suppose that* $(\Omega,\ \lambda,\ \mu,\ \rho)$ *(respectively,* $(\Omega,\ \widehat{\lambda},\ \widehat{\mu},\ \widehat{\rho})$*) are isotropic, linearly elastic media satisfying the conditions of Section A.1. In addition, suppose that*

$$\Lambda_{\mathbf{e},\lambda,\mu,\rho} \;=\; \Lambda_{\mathbf{e},\widehat{\lambda},\widehat{\mu},\widehat{\rho}} \quad \textit{on} \;\; (0,T),$$

where $T<\infty$ *is greater than the geodesic distance (with respect to each of* $g_p,\ g_s,\ \widehat{g}_p,\ \widehat{g}_s$*) between any two boundary points of* Ω*. Then for any* $X \in \bar{\Omega}$

$$\lambda(X) = \widehat{\lambda}(X), \qquad \mu(X) = \widehat{\mu}(X), \quad \textit{and} \qquad \rho(X) = \widehat{\rho}(X).$$

We denote by

$$\mathfrak{O}_{iso} \;=\; \{(\psi^*\mathbf{G}, \psi^*\mathbf{C}_{iso}, \psi^*\rho) \mid \psi \in \Psi\}$$

the orbit of isotropic, linearly elastic media. We now prove uniqueness for the anisotropic, linearly elastic media in the orbit $\mathfrak{O}_{iso}$ by applying Theorem 1, equation (12), and Theorem 3. (In Theorem 11 we give a pointwise characterization of $\mathfrak{O}_{iso}$ and a pointwise description of parts of the orbits of other types of anisotropic elastic media.)

Corollary 4 (Partial uniqueness within the orbit generated by isotropic elastic media)**.** *Consider admissible, linearly elastic media* $(\Omega, \psi^*\mathbf{G}, \psi^*\mathbf{C}_{iso}, \psi^*\rho)$, $(\Omega,\ \widehat{\psi}^*\widehat{\mathbf{G}},\ \widehat{\psi}^*\widehat{\mathbf{C}}_{iso},\ \widehat{\psi}^*\widehat{\rho}) \;\in\; \mathfrak{O}_{iso}$*. Suppose that the transformed media* $\big(\Omega, \mathbf{e}, \sqrt{\det\,\mathbf{G}}(\lambda,\mu,\rho)\big)$ *and* $\big(\Omega, \mathbf{e}, \sqrt{\det\,\widehat{\mathbf{G}}}(\widehat{\lambda},\widehat{\mu},\widehat{\rho})\big)$ *satisfy the conditions of Theorem 3. In addition, suppose that*

$$\Lambda_{\psi^*\mathbf{G},\psi^*\mathbf{C}_{iso},\psi^*\rho} \;=\; \Lambda_{\widehat{\psi}^*\widehat{\mathbf{G}},\ \widehat{\psi}^*\widehat{\mathbf{C}}_{iso},\ \widehat{\psi}^*\widehat{\rho}}\,. \tag{15}$$

Then for $X \in \overline{\Omega}$ *and* $\delta = \sqrt{\det\,\widehat{\mathbf{G}}}/\sqrt{\det\,\mathbf{G}}$

$$\lambda(X) \;=\; \delta\,\widehat{\lambda}(X), \qquad \mu(X) \;=\; \delta\,\widehat{\mu}(X), \quad \textit{and} \qquad \rho(X) \;=\; \delta\,\widehat{\rho}(X).$$

Proof. By Theorem 1 the left side $\Lambda_{\psi^*\mathbf{G},\psi^*\mathbf{C}_{iso},\psi^*\rho}$ of (15) may be written as $\Lambda_{\mathbf{G},\mathbf{C}_{iso},\rho}$, which by (12) agrees with $\Lambda_{\mathbf{e},\sqrt{\det\mathbf{G}}(\mathbf{C}_{iso},\rho)}$. Similarly, the right side of (15) may be written as $\Lambda_{\mathbf{e},\sqrt{\det\widehat{\mathbf{G}}}(\widehat{\mathbf{C}}_{iso},\widehat{\rho})}$. The result now follows by (15) and Theorem 3. $\square$

Remark 5. In particular, we may apply Corollary 4 to prove uniqueness for isotropic elastodynamics in the setting of any arbitrary, flat metric $\bar{\mathbf{G}}$ of the form $\psi^*\mathbf{e}$, $\psi \in \Psi$. In this case the elasticity tensor C (isotropic with respect to $\bar{\mathbf{G}}$) has the form

$$\mathsf{C}^{ABCD} \;=\; \lambda(X)\bar{G}^{AB}\bar{G}^{CD} \;+\; \mu(X)\big[\bar{G}^{AC}\bar{G}^{BD} \;+\; \bar{G}^{AD}\bar{G}^{BC}\big].$$

In fact, $\mathsf{C} = \psi^*\mathsf{C}_{iso}$.

We next recall Rachele's uniqueness result for linear, isotropic elastic media *with residual stress* in Euclidean space.

Theorem 6 (Rachele [R IV], Partial uniqueness for $(\lambda, \mu, \rho, \mathcal{T})$ in Euclidean space). *Suppose that* $(\Omega,\, \lambda,\, \mu,\, \rho,\, \mathcal{T})$ *(respectively,* $(\Omega,\, \widehat{\lambda},\, \widehat{\mu},\, \widehat{\rho},\, \widehat{\mathcal{T}})$*) are isotropic, linearly elastic media with residual stress as described in Section A.2. Let*

$$g_p \;=\; \Big(\frac{\lambda + 2\mu}{\rho} I + \frac{1}{\rho}\mathcal{T}\Big)^{-1} \qquad \textit{and} \qquad g_s \;=\; \Big(\frac{\mu}{\rho} I + \frac{1}{\rho}\mathcal{T}\Big)^{-1}$$

denote the metrics whose geodesics are the paths of wave propagation in these media (respectively, $\widehat{g}_p$ *and* $\widehat{g}_s$*). In addition, suppose that*

$$\Lambda_{\mathbf{e},\lambda,\mu,\rho,\mathcal{T}} \;=\; \Lambda_{\mathbf{e},\widehat{\lambda},\widehat{\mu},\widehat{\rho},\widehat{\mathcal{T}}} \quad \textit{on} \;\; (0, T),$$

where $T < \infty$ *is greater than the geodesic distance (with respect to each of* g_p, g_s, $\widehat{g}_p$, $\widehat{g}_s$*) between any two boundary points of* Ω*. Then*

$$\psi_p^* g_p \;=\; \widehat{g}_p \quad \textit{and} \quad \psi_s^* g_s \;=\; \widehat{g}_s \quad \textit{in} \;\; \Omega$$

for some diffeomorphisms ψ_p, ψ_s *of* $\bar{\Omega}$ *which are the identity on* $\partial\Omega$.

We again apply Theorem 1, equation (12), and Theorem 6, here to prove uniqueness for the anisotropic (with respect to $\mathbf{e}$) elastic media in the orbit

$$\mathfrak{O}_{resid} \;=\; \{\; (\psi^*\mathbf{G}, \psi^*\mathsf{C}_{iso}, \psi^*\rho, \psi^*\mathcal{T}) \;|\; \psi \in \Psi \;\}$$

of $\mathbf{e}$-isotropic elastic media with residual stress. The equations of motion for such media are given by (36) with second Piola-Kirchhoff stress tensor $\mathbf{S}$ given by the residual stress tensor $\mathcal{T}$.

Corollary 7 (Partial uniqueness within the orbit generated by isotropic elastic media with residual stress). *Consider admissible linearized elastic media* $(\Omega, \psi^*\mathbf{G}, \psi^*\mathsf{C}_{iso}, \psi^*\rho, \psi^*\mathcal{T})$, $(\Omega,\, \widehat{\psi}^*\widehat{\mathbf{G}},\, \widehat{\psi}^*\widehat{\mathsf{C}}_{iso},\, \widehat{\psi}^*\widehat{\rho},\, \widehat{\psi}^*\widehat{\mathcal{T}}) \in \mathfrak{O}_{resid}$ *with the property that the transformed media* $\big(\Omega, \mathbf{e}, \sqrt{\det\, \mathbf{G}}(\lambda, \mu, \rho, \mathcal{T})\big)$ *and* $\big(\Omega, \mathbf{e}, \sqrt{\det\, \widehat{\mathbf{G}}}(\widehat{\lambda}, \widehat{\mu}, \widehat{\rho}, \widehat{\mathcal{T}})\big)$ *satisfy the conditions of Theorem 6. In addition, suppose that*

$$\Lambda_{\psi^*\mathbf{G},\psi^*\mathsf{C}_{iso},\psi^*\rho,\psi^*\mathcal{T}} \;=\; \Lambda_{\widehat{\psi}^*\widehat{\mathbf{G}},\, \widehat{\psi}^*\widehat{\mathsf{C}}_{iso},\, \widehat{\psi}^*\widehat{\rho},\, \widehat{\psi}^*\widehat{\mathcal{T}}}\,.$$

Then for g_p, g_s, $\widehat{g}_p$, $\widehat{g}_s$ *as defined in Theorem 6*

$$\psi_p^* g_p \;=\; \widehat{g}_p \quad \textit{and} \quad \psi_s^* g_s \;=\; \widehat{g}_s \quad \textit{in} \;\; \Omega$$

for some diffeomorphisms ψ_p, ψ_s *of* $\bar{\Omega}$ *which are the identity on* $\partial\Omega$.

Corollaries 4 and 7 illustrate an approach that may be taken to prove partial uniqueness for more general anisotropic (linear or linearized) elastic media. The strategy is to prove uniqueness for the simplest class of anisotropic media in an orbit $\mathfrak{O}$, and then to derive partial uniqueness, via Theorem 1, for other anisotropic media in $\mathfrak{O}$.

2.2. **The Symmetry Group of $\psi^*\mathbf{C}$.** In the previous section we extend uniqueness results for elastic media to their orbits under the action of Ψ. Consequently, we are interested in characterizing the orbits of each type of (an)isotropic elastic media, especially the orbit of isotropic elastic media (for which uniqueness results are already known). In this section we characterize the orbit of isotropic elastic media. (At any $X \in \Omega$ the orbit of the isotropic elasticity tensors consists of the isotropic, some transverse isotropic, and some orthotropic elasticity tensors.) We also describe $\psi^*\mathbf{C}(X)$ for most other types of anisotropic media, for certain $\psi \in \Psi$. In particular, we describe the type of anisotropy of $\psi^*\mathbf{C}$ via its material symmetry group with respect to the Euclidean metric.

The *material symmetry group* $\mathcal{G}$ associated with an elastic medium is defined to be the subgroup of $\mathbf{G}$-orthogonal transformations $Q : T_X\Omega \to T_X\Omega$ (i.e. with $Q^T = Q^{-1}$, where, in coordinates, $Q^T(X)^A{}_B = G_{BB'}Q^{B'}{}_{A'}G^{A'A}$) whose action on the reference configuration does not change the material response. In particular, the symmetry group $\mathrm{Sym}_{\mathbf{e}}\mathbf{T}$ at $X \in \Omega$ of an m-tensor field $\mathbf{T}$ on Ω (e.g. an elasticity tensor $\mathbf{C}$) is defined to be the collection of $\mathbf{e}$-orthogonal transformations Q of $T_X\Omega$ (i.e. with $Q^T(X)^A{}_B = \delta_{BB'}Q^{B'}{}_{A'}\delta^{A'A}$) such that

$$Q * \mathbf{T}(X) = \mathbf{T}(X),$$

where for any 3×3 matrix M (which we identify with a $(1,1)$-tensor) we define $M * \mathbf{T}(X)$ to be the m-tensor with components

$$(M * \mathbf{T}(X))^{ABC\ldots D} = M^A{}_{\underline{A}}\, M^B{}_{\underline{B}}\, M^C{}_{\underline{C}} \ldots M^D{}_{\underline{D}}\, \mathsf{T}(X)^{\underline{A}\,\underline{B}\,\underline{C}\ldots\underline{D}}. \tag{16}$$

We remark that the $*$ operation respects the symmetries of $\mathbf{T}$ for any invertible matrix M; in particular, it defines an action of the orthogonal group $O(3)$ on the space of elasticity tensors.

In addition, the $*$ operation defines an action of $O(3)$ on the symmetry groups of elasticity tensors. In fact, the action of $Q \in O(3)$ on the symmetry group of a tensor $\mathbf{T}$ is given by the symmetry group of $Q * \mathbf{T}$:

$$\mathrm{Sym}_e(Q * \mathbf{T}) = Q(\,\mathrm{Sym}_e\mathbf{T})Q^T = Q * (\,\mathrm{Sym}_e\mathbf{T}) \tag{17}$$

since $\overline{Q} \in \mathrm{Sym}_e\mathbf{T}$ if and only if $Q\overline{Q}Q^T * (Q * \mathbf{T}) = Q * \mathbf{T}$. Here we write $M * G = MGM^T$ in terms of conjugation by M for any rank-2 tensor G, and we write

$$M * \mathcal{G} = M\,\mathcal{G}\,M^T = \{MGM^T \mid G \in \mathcal{G}\,\} \tag{18}$$

TABLE 1. **Classification of Anisotropic Elastic Media**

i	Symmetry Class	Generators of $SO(3) \cap$ the symmetry group $\mathcal{G}_i$
1	triclinic	I
2	monoclinic	$Q^{\pi}_{e_3}$ (cyclic group $\mathcal{Z}_2$)
3	orthotropic	$Q^{\pi}_{e_1}, Q^{\pi}_{e_2}, Q^{\pi}_{e_3}$ (dihedral group $\mathcal{D}_2$)
4	trigonal	$Q^{\pi}_{e_1}, Q^{2\pi/3}_{e_3}$ (dihedral group $\mathcal{D}_3$)
5	tetragonal	$Q^{\pi}_{e_1}, Q^{\pi}_{e_2}, Q^{\pi/2}_{e_3}$ (dihedral group $\mathcal{D}_4$)
6	cubic	$Q^{\pi/2}_{e_1}, Q^{\pi/2}_{e_2}, Q^{\pi/2}_{e_3}$ (octahedral group $\mathcal{O}$)
7	transversely isotropic	$Q^{\pi}_{e_1}, Q^{\pi}_{e_2}, Q^{\phi}_{e_3}$ (orthogonal group $O(2)$)
8	isotropic	(special orthogonal group $SO(3)$)

for any collection of linear transformations $\mathcal{G}$.

Given an elasticity tensor $\mathbf{C}$ with a certain type of anisotropy (one of eight types, given [F-V]), there exists a right-handed orthonormal basis $\{e_1, e_2, e_3\}$ so that the symmetry group of $\mathbf{C}$ is generated by $-I$ and the orthogonal transformations listed in Table 1. (See [C-V-C], for example.) The generators in Table 1 have the form $Q^{\phi}_{e_j}$, $j = 1, 2, 3$, where $Q^{\phi}_{e_j}$ is the rotation of angle ϕ about e_j; e.g., $Q^{\phi}_{e_3}(xe_1+ye_2+ze_3) = [x\cos\phi - y\sin\phi]e_1 + [y\cos\phi + x\sin\phi]e_2 + ze_3$. Throughout, $\mathrm{sg}\{Q^{\pi}_{e_i}, Q^{\pi}_{e_j}\}$ denotes the subgroup of $O(3)$ generated by $Q^{\pi}_{e_i}, Q^{\pi}_{e_j}$. These generators are not independent in the sense that, for example, $\mathrm{sg}\{Q^{\pi}_{e_1}, Q^{\pi}_{e_2}\} = \mathrm{sg}\{Q^{\pi}_{e_1}, Q^{\pi}_{e_2}, Q^{\pi}_{e_3}\}$ since $Q^{\pi}_{e_1} \circ Q^{\pi}_{e_2}(xe_1 + ye_2 + ze_3) = Q^{\pi}_{e_1}(-xe_1 + ye_2 - ze_3) = (-xe_1 - ye_2 + ze_3) = Q^{\pi}_{e_3}(xe_1 + ye_2 + ze_3)$.

Two elasticity tensors $\mathbf{C}_1$ and $\mathbf{C}_2$ have the same anisotropy type at $X \in \Omega$ if and only if there exists an orthogonal transformation Q such that

$$\mathbf{C}_1(X) \;=\; Q * \mathbf{C}_2(X).$$

To describe the symmetry group of $\psi^*\mathbf{C}$ for $\psi \in \Psi$ at a fixed point $X \in \Omega$, we write $D\psi^{-1}(X)$ in terms of its polar decomposition with the positive, symmetric S diagonalized as $S = UDU^T$. Here U, V are orthogonal, and $D = \mathrm{diag}\,(a, b, c)$ is a diagonal 3×3 matrix with a, b, c the (positive) eigenvalues of S. Without loss of generality, we assume either that $a = b = c$, that $a = b \neq c$, or that the a, b, c are distinct.

We observe that at $X \in \Omega$

$$\psi^*\mathbf{C} = (D\psi^{-1}) * (\mathbf{C} \circ \psi) = UDU^TV * (\mathbf{C} \circ \psi) = UD * \bar{\mathbf{C}} = U * \widehat{\bar{\mathbf{C}}}, \tag{19}$$

where

$$\bar{\mathbf{C}} = U^T V * (\mathbf{C} \circ \psi) \qquad \text{and} \qquad \widehat{\bar{\mathbf{C}}} = D * \bar{\mathbf{C}}. \tag{20}$$

It follows, given (17), that we may reduce the task of describing $\mathrm{Sym}_e \psi^* \mathbf{C}$ to that of describing $\mathrm{Sym}_e(D * \bar{\mathbf{C}})$ for any elasticity tensor $\bar{\mathbf{C}}$.

Proposition 8. *Let $\mathbf{C}$ be an admissible elasticity tensor on Ω. For $X \in \Omega$ and $\psi \in \Psi$ with polar decomposition at X in the form described above*

$$\mathrm{Sym}_e \psi^* \mathbf{C} = U * \mathrm{Sym}_e(D * \bar{\mathbf{C}}), \tag{21}$$

$$\mathrm{Sym}_e \bar{\mathbf{C}} = U^T V * \mathrm{Sym}_e(\mathbf{C} \circ \psi). \tag{22}$$

We note that the action on $\mathrm{Sym}_e \bar{\mathbf{C}}$ of the pullback by D is accomplished at each point $X \in \Omega$ by varying only two parameters a/c and b/c.

A partial description of the relationship at $X \in \Omega$ between $\mathrm{Sym}_{\mathbf{e}} \bar{\mathbf{C}}$ and $\mathrm{Sym}_{\mathbf{e}}(D * \bar{\mathbf{C}})$ is given in the following proposition.

Proposition 9. *Let $\bar{\mathbf{C}}$ be an admissible elasticity tensor on Ω. Let D be a 3×3 matrix. Then*

$$\mathrm{Sym}_{\mathbf{e}} \bar{\mathbf{C}} \ \cap \ \mathrm{Sym}_e D \quad \subseteq \quad \mathrm{Sym}_{\mathbf{e}}(D * \bar{\mathbf{C}}). \tag{23}$$

Proof. For $Q \in \mathrm{Sym}_{\mathbf{e}} \bar{\mathbf{C}} \cap \mathrm{Sym}_{\mathbf{e}} D$ we have $Q * (D * \bar{\mathbf{C}}) = QD * \bar{\mathbf{C}} = DQ * \bar{\mathbf{C}} = D * \bar{\mathbf{C}}$. That is, $Q \in \mathrm{Sym}_{\mathbf{e}}(D * \bar{\mathbf{C}})$. □

The simplest situation in (23), and the one we consider here, occurs when S is diagonal with respect to the basis $\{e_1, e_2, e_3\}$ used to describe $\mathrm{Sym}_e V * (\bar{\mathbf{C}} \circ \psi)$. The computational complexity of the other cases is significantly greater. This motivates the following definition. We say that ψ is *aligned with* $\mathbf{C}$ (at $X' \in \Omega$) if there is a polar decomposition of $D\psi^{-1}$ and diagonalization of the stretch factor S as described above so that, with respect to the standard basis for Euclidean space, the generators of $\mathrm{Sym}_e \bar{\mathbf{C}}(X')$ are listed in Table 1. We observe below that for certain aligned ψ and $\mathbf{C}$ equality holds in (23).

Remark 10. For $\mathbf{C}$ isotropic, every $\psi \in \Psi$ is aligned with $\mathbf{C}$.

Without loss of generality, we set $c = 1$ below.

Theorem 11 (Computing $D * \bar{\mathbf{C}}$ in the aligned case)**.** *Let $\bar{\mathbf{C}}$ be an admissible isotropic, transverse isotropic, cubic, or tetragonal elasticity tensor on Ω represented by parameters $(\bar{\alpha}, \bar{\beta}, \bar{A}^{11}, \bar{B}^{11}, \bar{\alpha}_0, \bar{\alpha}_4)$ via the harmonic-Cartan decomposition (51) and (56). Suppose that, with respect to the standard basis for Euclidean space, the generators of $\mathrm{Sym}_e \bar{\mathbf{C}}$ are listed in Table 1. Let $D = \mathrm{diag}\,(a, b, 1)$ be a diagonal 3×3 matrix with a, b positive and $a = b \neq 1$ or $a, b, 1$ distinct. Then via the harmonic-Cartan decomposition we may represent $D * \bar{\mathbf{C}}$ by $(\widehat{\bar{\alpha}}, \widehat{\bar{\beta}}, \widehat{\bar{A}}^{11}, \widehat{\bar{B}}^{11}, \widehat{\bar{\alpha}}_0, \widehat{\bar{\alpha}}_2, \widehat{\bar{\alpha}}_4)$, and the (scaled) parameters $(15\widehat{\bar{\alpha}}, 15\widehat{\bar{\beta}}, 21\widehat{\bar{A}}^{11}, 21\widehat{\bar{A}}^{22}, 21\widehat{\bar{B}}^{11}, 21\widehat{\bar{B}}^{22}, 280\widehat{\bar{\alpha}}_0, 14\widehat{\bar{\alpha}}_2, 8\widehat{\bar{\alpha}}_4)$ for $D * \bar{\mathbf{C}}$ are given by*

$$\begin{aligned}\bar{\alpha}\cdot P_{\bar{\alpha}}(a,b) \;+\; \bar{\beta}\cdot P_{\bar{\beta}}(a,b) \;+\; \bar{A}^{11}\cdot P_{\bar{A}^{11}}(a,b)\\ +\; \bar{B}^{11}\cdot P_{\bar{B}^{11}}(a,b) \;+\; \bar{\alpha}_0\cdot P_{\bar{\alpha}_0}(a,b) \;+\; \bar{\alpha}_4\cdot P_{\bar{\alpha}_4}(a,b),\end{aligned} \tag{24}$$

where the $P(a,b)$, *defined by* (73), *are linearly independent for fixed* a,b.

In fact, $\mathrm{Sym}_e\bar{\mathbf{C}} \not\subseteq \mathrm{Sym}_e D$ *implies*

$$\mathrm{Sym}_e\bar{\mathbf{C}} \;\cap\; \mathrm{Sym}_e D \;=\; \mathrm{Sym}_e(D*\bar{\mathbf{C}}). \tag{25}$$

Remark 12. Note that in the case that $a=b=c$ we have that $D*\bar{\mathbf{C}} = a^4\bar{\mathbf{C}}$, and so the pullback by ψ preserves the type of anisotropy of the elasticity tensor.

The proof of Theorem 11, given in Section 5, is based on the harmonic decomposition (51) of the elasticity tensor $\bar{\mathbf{C}}$, followed by the Cartan decomposition (56) of the fourth-order, harmonic part $\bar{\mathbf{H}}$ of $\bar{\mathbf{C}}$, as described in [F-V]. Due to this decomposition we may write exact formulas for the (an)isotropic elasticity tensors $\bar{\mathbf{C}}$ and $D*\bar{\mathbf{C}}$. We then use the mathematical software Maple to compute the harmonic-Cartan decomposition of $D*\bar{\mathbf{C}}$, and analyze the (possible) vanishing of certain terms in order to relate $\mathrm{Sym}_e\bar{\mathbf{C}}$ with $\mathrm{Sym}_e D*\bar{\mathbf{C}}$.

Below we relate the type of (an)isotropy in the aligned case of isotropic, transversely isotropic, cubic, and tetragonal $\bar{\mathbf{C}}$ and the corresponding $D*\bar{\mathbf{C}}$.

Corollary 13 (Characterizing the range of the $D*$ operation on some aligned $\bar{\mathbf{C}}$)**.** *Suppose that* $\bar{\mathbf{C}}$ *satisfies the conditions of Theorem 11. Then:*

- i). *For* $\bar{\mathbf{C}}$ *isotropic and* $a=b\neq 1$, *the range of* $D*\bar{\mathbf{C}}$ *is a 3-parameter family of the form* $\bar{\alpha}\cdot P_{\bar{\alpha}}(a,b)+\bar{\beta}\cdot P_{\bar{\beta}}(a,b)$ *in the 5-parameter family of transverse isotropic elasticity tensors.*
- ii). *For* $\bar{\mathbf{C}}$ *cubic and* $a=b\neq 1$, *the range of* $D*\bar{\mathbf{C}}$ *is a 4-parameter family of the form* $\bar{\alpha}\cdot P_{\bar{\alpha}}(a,b)+\bar{\beta}\cdot P_{\bar{\beta}}(a,b)+\bar{\alpha}_0\cdot[P_{\bar{\alpha}_0}(a,b)+5P_{\bar{\alpha}_4}(a,b)]$ *in the 5-parameter family of tetragonal elasticity tensors.*
- iii). *If* $a,b,1$ *are distinct, then for isotropic, transverse isotropic, cubic, or tetragonal* $\bar{\mathbf{C}}$, *the range of* $D*\bar{\mathbf{C}}$ *is a family of orthotropic elasticity tensors of the form (24).*

Corollary 13 is a consequence of Theorem 11 and Section 5.2, and is derived in Section 5.4. The following proposition is also proved in Section 5.4.

Proposition 14. *If* $\mathbf{C}$ *is isotropic, transverse isotropic, or orthotropic, and* ψ *is aligned with* $\mathbf{C}$ *at* $X'\in\Omega$, *then* ψ^{-1} *is aligned with* $\psi^*\mathbf{C}$ *at* $X=\psi(X')$.

Corollary 15 ($D*$ preserves transverse isotropy for some aligned $\bar{\mathbf{C}}(X)$)**.** *For* $X\in\Omega$ *there is a(n at least two-parameter) non-trivial family of transverse isotropic* $\bar{\mathbf{C}}(X)$ *with the property that* $D*\bar{\mathbf{C}}(X)$ *remains transverse isotropic when* $a=b\neq 1$.

3. Definitions

We follow Marsden and Hughes [M-H] to represent an elastic body as a Riemannian manifold $(\Omega, \mathbf{G})$. Let $(\mathcal{S}, \mathbf{g})$ be a (fixed) Riemannian manifold of the same dimension representing the ambient space within which the body Ω moves. In this paper $\mathcal{S} = \mathbb{R}^3$ and for applications we take $\mathbf{g}$ to be the Euclidean metric. We write $\{X^A\}$ for coordinate systems on Ω, and $\{x^a\}$ for coordinate systems on the ambient space $\mathcal{S}$.

General tensor notation. We follow notational conventions for tensor analysis as in [M-H]. In particular, tensor fields are generally written in boldface and their components in standard font, e.g. the so-called deformation gradient $\mathbf{F}$ is a tensor field with components $\mathrm{F}^a{}_A$.

A two-point tensor $\mathbf{T}$ of type $\begin{pmatrix} q & l \\ p & m \end{pmatrix}$ at $X \in \Omega$ over a mapping $\phi : \Omega \to \mathcal{S}$ is a multilinear mapping into $\mathbb{R}$ from the product space consisting of p copies of $T^*_X\Omega$, q copies of $T_X\Omega$, l copies of $T^*_x\mathcal{S}$, and m copies of $T_x\mathcal{S}$, where $x = \phi(X)$. The components of $\mathbf{T}$ are denoted by $T^{A_1 \dots A_p}{}_{B_1 \dots B_q}{}^{a_1 \dots a_l}{}_{b_1 \dots b_m}$, where, in general, upper-case indices denote tensorial behavior with respect to the referential setting, lower-case indices with respect to the spatial setting, and the pullback of a two-point tensor by $\psi : \Omega \to \Omega$ interacts only with indices corresponding to the reference configuration (i.e. the upper-case indices). We use the terminology *two-point vector* for a two-point tensor of type $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

The non-linear case. The deformation of an elastic body is modeled by mappings of the form $\phi(X,t) = \phi_t(X) : \Omega \times (0,T) \to \mathcal{S}$. In fact, we call a mapping $\phi : \Omega \to \mathcal{S}$ a *configuration* of Ω in $\mathcal{S}$, and we denote by $\mathcal{C}$ the set of all configurations of Ω in $\mathcal{S}$. A *motion* $\phi_t : \Omega \to \mathcal{S}$ *of the body* Ω *in* $\mathcal{S}$ is a curve in $\mathcal{C}$ parametrized by t, where t ranges over some open interval of $\mathbb{R}$. A motion ϕ_t is C^2 *regular* if ϕ_t is invertible on $\phi_t(\Omega)$ for each t, and if $\phi_t(X)$ and $\phi_t^{-1}(X)$ are C^2 as functions of t and $X \in \Omega$. We write $x = \phi_t(X)$.

For any fixed diffeomorphism $\psi \in \Psi$ of $\bar{\Omega}$ we write

$$\phi'_t = \psi^* \phi_t, \qquad X' = \psi^{-1}(X), \qquad \mathbf{G}' = \psi^* \mathbf{G},$$

for any ϕ_t that is a C^2 regular motion of Ω in $\mathcal{S}$, where $t \in [0,T]$ with $0 < T < \infty$. We remark that ϕ'_t is also a C^2 regular motion of Ω in $\mathcal{S}$.

The *Cauchy stress tensor* $\boldsymbol{\sigma}(x,t)$ is a $\binom{2}{0}$ tensor field which has the property that the *Cauchy stress vector* $\mathbf{t}(x,t,\mathbf{n}) = \boldsymbol{\sigma} \cdot \mathbf{n}$ represents the force per unit (of deformed) area exerted on a surface element that is oriented with unit normal vector $\mathbf{n}(\mathrm{x})$. The first Piola-Kirchhoff stress tensor $\mathbf{P}(X,t)$ is the Piola transform (on the second index) of the Cauchy stress tensor $\boldsymbol{\sigma}$; that is,

$$P^{aA}(X) \;=\; J(X,t)\; (\mathrm{F}^{-1})^A{}_b\; \sigma^{ab}, \tag{26}$$

where

$$\mathbf{F}(X,t) \;=\; D\phi_t(X), \quad \text{with } \mathrm{F}(X,t)^a{}_A \;=\; D\phi_t(X)^a{}_A \;=\; \frac{\partial {\phi_t}^a}{\partial X^A} \tag{27}$$

is the deformation gradient (with respect to a motion ϕ_t of Ω), and the scalar

$$J(X,t) \;=\; \frac{\sqrt{\det\, g}}{\sqrt{\det\, G}}\,\det\,(D\phi_t) \tag{28}$$

is the Jacobian of the linear transformation $\mathbf{F}$.

The second Piola-Kirchhoff stress tensor $\mathbf{S}(X,t)$ is the pull-back of the first leg of $\mathbf{P}$ via ϕ_t, i.e. $\mathrm{S}^{\mathrm{AB}} = (\mathrm{F}^{-1})^{\mathrm{A}}{}_{\mathrm{a}}\mathrm{P}^{\mathrm{aB}}$.

A material is called *hyperelastic* if there exists a scalar functional $W = \widehat{W}(X,\mathbf{G},\mathbf{C}^\flat)$ such that

$$\widehat{S}^{AB} \;=\; 2\rho\,\frac{\partial \widehat{W}}{\partial {C^\flat}_{AB}},$$

where $\mathbf{C}^\flat = \phi_t^* g$ is related to the *deformation tensor* $\mathbf{C} = \mathbf{F}^T\mathbf{F}$ by $C^A{}_B = G^{A\underline{A}}{C^\flat}_{\underline{A}B}$. W is called the *stored energy function*, and corresponds to the internal energy per unit mass $E = \widehat{E}(X,\mathbf{G},\mathbf{C}^\flat)$ at thermal equilibrium.

It follows from the constitutive equations that the elasticity tensor $\mathsf{C} = \widehat{\mathsf{C}}(X,\mathbf{G},\mathbf{C}^\flat)$, which measures the dependence of stress on the strain, may be written as

$$\mathsf{C}^{ABCD} \;=\; 2\rho\;\frac{\partial^2 \widehat{E}}{\partial {C^\flat}_{AB}\partial {C^\flat}_{CD}}. \tag{29}$$

That is, in this case C depends on the motion ϕ_t, and so we view $\mathbf{G}$ and $\mathcal{O} = (E,\rho)$ as the material parameters of the elastic object.

We say that ϕ_t *solves the initial-boundary-value problem for non-linear hyperelasticity* if, for given Dirichlet boundary data f,

$$F_{\mathbf{G},E,\rho}(\phi_t) \;=\; \rho\mathbf{A} \;-\; \mathrm{DIV}_{\mathbf{G}}\,\mathbf{P} \;=\; 0 \qquad \text{on } \Omega, \tag{30a}$$

$$\phi_t(X) \;=\; X + f(X,t) \qquad \text{for } X \in \partial\Omega, \tag{30b}$$

$$\phi_t(X) \;=\; 0 \qquad \text{for } t = 0, \tag{30c}$$

$$\mathbf{V}(X,t) \;=\; 0 \qquad \text{for } t = 0, \tag{30d}$$

where $\mathbf{V}(X,t) = \partial\phi_t/\partial t$ is the material velocity vector field, and $\mathbf{A}(X,t)$, the acceleration vector field, is given by

$$A^a = \partial V^a/\partial t + (\gamma^a_{bc}\circ\phi_t)V^bV^c \tag{31}$$

on Ω. Throughout, γ^a_{bc} and Γ^A_{BC} denote (respectively) the Christoffel symbols for the spatial metric $\mathbf{g}$ and referential metric $\mathbf{G}$. The divergence of a two-point tensor $\mathbf{P} = (P^{aA})$ is defined in the referential setting by

$$(\mathrm{DIV}_X\,\mathbf{P})^a \;=\; \frac{\partial P^{aA}}{\partial X^A} \;+\; \Gamma^{\underline{A}}_{\underline{A}A}P^{aA} \;+\; (\gamma^a_{\underline{b}\,\underline{a}}\circ\phi)\mathrm{F}^{\underline{b}}{}_A P^{\underline{a}A}, \tag{32}$$

as in [M-H].

We refer to [M-H], p. 401, 406, for a discussion of existence and uniqueness for the non-linear initial-boundary-value problem.

The surface measurements used to determine the material parameters $\mathbf{G}$, E, ρ of the elastic object are modeled by the dynamic Dirichlet-to-Neumann map $\Lambda_{\mathbf{G},E,\rho}$. The Dirichlet-to-Neumann map gives the correspondence (when there exist unique solutions of (30)) between the surface displacement and the surface traction taken over a finite period of time. Following [Sy-U], we view the Dirichlet-to-Neumann map as a vector-valued form representing the flux, given in the case of non-linear elasticity by

$$(\Lambda_{\mathbf{G},E,\rho} f\,)^a \;=\; \langle \mathbf{P},\, \boldsymbol{\nu}_{\mathbf{G}} \rangle^a \; dS_{\mathbf{G}} \;=\; P^{aA}\; \mathbf{G}_{AB}\; \boldsymbol{\nu}^B_{\mathbf{G}}\; dS_{\mathbf{G}} \tag{33}$$

for $t \in [0,T]$, where $dS_{\mathbf{G}}$ is the area form on $\partial\Omega$ induced by $\mathbf{G}$, $\boldsymbol{\nu}_{\mathbf{G}}$ is the unit outer normal to $\partial\Omega$ with respect to $\mathbf{G}$, and the motion ϕ_t solves (30) for boundary values f. By Corollary 2 the Dirichlet-to-Neumann map may also be written with respect to the Euclidean metric as

$$(\Lambda_{\mathbf{G},E,\rho} f\,)^a \;=\; P^{aA}\; \delta_{AB}\; \boldsymbol{\nu}^B\; \sqrt{\det\, \mathbf{G}}\;\, dS, \tag{34}$$

where dS is the area form on $\partial\Omega$, and $\boldsymbol{\nu}$ is the unit outer normal vector to $\partial\Omega$, each with respect to the Euclidean metric.

The linearized case. We linearize about an arbitrary C^2-regular motion $\mathring{\phi}_t$ which solves (30a), as in [M-H], Section 4.2. The initial-boundary-value problem in this case is given by

$$
\begin{aligned}
&\text{(35a)} \qquad & P_{\mathbf{G},\mathring{\mathrm{A}},\mathring{\mathbf{C}},\rho,\mathring{\mathbf{S}},\mathring{\mathbf{P}}}\,\mathbf{U} &= 0 && \text{on } \Omega,\\
&\text{(35b)} & \mathbf{U}(X,t) &= f(X,t) && \text{for } X \in \partial\Omega,\\
&\text{(35c)} & \mathbf{U}(X,t) &= 0 && \text{for } t = 0,\\
&\text{(35d)} & \partial \mathbf{U}(X,t)/\partial t &= 0 && \text{for } t = 0,
\end{aligned}
$$

with Dirichlet boundary data f, where the operator $P_{\mathbf{G},\mathring{\mathrm{A}},\mathring{\mathbf{C}},\rho,\mathring{\mathbf{S}},\mathring{\mathbf{P}}}$ is given by

$$P_{\mathbf{G},\mathring{\mathrm{A}},\mathring{\mathbf{C}},\rho,\mathring{\mathbf{S}},\mathring{\mathbf{P}}}\,\mathbf{U} \;=\; \rho\Big(\frac{\partial^2 \mathbf{U}(X,t)}{\partial t^2} \;+\; \mathring{\mathbf{A}}\Big) - \;\mathrm{DIV}_{\mathbf{G}}(\mathring{\mathbf{A}} \cdot \nabla \mathbf{U} \;+\; \mathring{\mathbf{P}}). \tag{36}$$

Here $\rho(X)$ is the mass density in the reference configuration; $\mathbf{U}$, the *infinitesimal displacement*, is a vector field over the motion $\mathring{\phi}_t$ (that is, a two-point vector over $\mathring{\phi}_t$); and $\mathring{\mathbf{A}}$, the first elasticity tensor, is given by

$$\mathring{\mathrm{A}}^{aA}{}_b{}^B = 2\mathring{\mathrm{C}}^{EAFB}\mathring{\mathrm{F}}^a{}_E\mathring{\mathrm{F}}^c{}_F\mathring{g}_{bc} + \mathring{\mathrm{S}}^{AB}\delta^a{}_b.$$

In addition, $\mathring{\mathbf{A}}(X,t)$ is the material acceleration, $\mathring{\mathbf{C}}$ is the (second) elasticity tensor defined in (29), $\mathring{\mathbf{F}}$ is the deformation gradient, and $\mathring{\mathbf{P}}$ and $\mathring{\mathbf{S}}$ are the first and second Piola-Kirchhoff stress tensors, respectively, all associated with $\mathring{\phi}_t$.

We emphasize that $\mathbf{U}$ does not behave like a vector field on Ω; in fact, the covariant derivative ∇ (w.r.t $\mathbf{G}$) of the two-point vector $\mathbf{U}$ over $\mathring{\phi}_t$ is given by

$$(\nabla \mathbf{U})^b{}_B = \partial \mathbf{U}^b / \partial X^B + (\mathring{\gamma}^b_{ac} \circ \mathring{\phi}_t)\, \mathbf{U}^a\, \mathrm{F}^c{}_B. \tag{37}$$

The Dirichlet-to-Neumann map is defined in this case by

$$\begin{aligned} (\Lambda_{\mathbf{G},\mathring{\mathsf{A}},\mathring{\mathsf{C}},\rho,\mathring{\mathbf{S}},\mathring{\mathbf{P}}} f\,)^a \;&=\; \big\langle \mathring{\mathsf{A}} \cdot \nabla \mathbf{U} \;+\; \mathring{\mathbf{P}},\; \boldsymbol{\nu} \big\rangle^a \; dS_{\mathbf{G}} \\ &=\; \big[\mathring{\mathsf{A}}^{aA}{}_b{}^B\; (\nabla \mathbf{U})^b{}_B \;+\; \mathring{\mathrm{P}}^{aA}\big]\; G_{AC}\; \nu^C_{\mathbf{G}}\; dS_{\mathbf{G}} \end{aligned} \tag{38}$$

for $t \in [0,T]$, where, again, $dS_{\mathbf{G}}$ is the area form on $\partial\Omega$ induced by $\mathbf{G}$, $\boldsymbol{\nu}_{\mathbf{G}}$ is the unit outer normal to $\partial\Omega$ with respect to $\mathbf{G}$, and $\mathbf{U}$ solves (35) with boundary data f.

By Corollary 2 the Dirichlet-to-Neumann map may also be written with respect to the Euclidean metric as

$$(\Lambda_{\mathbf{G},\mathring{\mathsf{A}},\mathring{\mathsf{C}},\rho,\mathring{\mathbf{S}},\mathring{\mathbf{P}}} f\,)^a \;=\; \big[\mathring{\mathsf{A}}^{aA}{}_b{}^B\; (\nabla \mathbf{U})^b{}_B \;+\; \mathring{\mathrm{P}}^{aA}\big]\;\; \delta_{AC}\;\; \nu^C\;\; \sqrt{\det \mathbf{G}}\;\; dS, \tag{39}$$

where, again, dS is the area form on $\partial\Omega$, and $\boldsymbol{\nu}$ is the unit outer normal vector to $\partial\Omega$, each with respect to the Euclidean metric.

The classical linear case. Linearizing about a stress-free, undeformed state $\mathring{\phi}(X,t)$, which we choose to be an embedding $\iota : \Omega \hookrightarrow \mathbb{R}^3$, results in the classical linear equations (4) and the associated Dirichlet-to-Neumann map (10). (See (6) and (12) for their representation in Euclidean coordinates.) In fact, in this case, $\mathring{\mathbf{S}}$, $\mathring{\mathbf{P}}$, and $\mathring{\mathbf{A}}$ vanish, so that (36) reduces to

$$P_{\mathbf{G},\mathsf{C},\rho}\,\mathbf{U} \;=\; \rho \frac{\partial^2 \mathbf{U}}{\partial t^2} \;-\; \mathrm{DIV}_{\mathbf{G}}(\mathsf{A} \cdot \nabla \mathbf{U}).$$

If the spatial metric $\mathbf{g}$ is Euclidean, as we take in applications, then ∇ acts on $\mathbf{U}$ as if $\mathbf{U}$ were a scalar, and the equations of motion, in local coordinates, are

$$\rho \frac{\partial^2 U^a}{\partial t^2} \;-\; \frac{1}{\sqrt{\det \mathbf{G}}}\; \frac{\partial}{\partial X^A} \Big(\sqrt{\det \mathbf{G}}\;\; \mathsf{A}^{aA}{}_b{}^B \frac{\partial U^b}{\partial X^B} \Big) \;=\; 0.$$

where A depends on the elasticity tensor C in this case simply via $\mathsf{A}^{aA}{}_b{}^B = 2\mathsf{C}^{EAFB} (D\iota)^a{}_E (D\iota)^c{}_F \delta_{bc}$.

Given any smooth diffeomorphism ψ of Ω, we observe that $\mathring{\phi}'(X,t) = \mathring{\phi}(\psi(X),t)$ is also stress-free, since the stress tensor $\mathbf{S}$ associated with a motion is covariant (see Section 4), so $\mathring{\mathbf{S}}' = \psi^* \mathring{\mathbf{S}} = 0$ since $\mathring{\mathbf{S}} = 0$.

It follows that when the equations of motion are rewritten in Euclidean coordinates, there are no "lower-order" terms arising from the Christoffel symbols, and so the equations of motion continue to describe an elastic response, but with respect to modified physical parameters. Therefore, as in the analysis of the wave equation, we may reduce anisotropic elasticity to the isotropic case by pulling back by ψ, at least for certain anisotropic media.

4. Proof of Theorem 1

We begin with **the non-linear case.** We model the behavior of the hyperelastic body $(\Omega, \mathbf{G}, E, \rho)$ by the initial-boundary-value problem (30). We will first show that the equations of motion (30a) are covariant; that is, given a sufficiently regular motion ϕ_t and any diffeomorphism $\psi : \bar{\Omega} \to \bar{\Omega}$, we show that

$$\psi^*(F_{\mathbf{G},E,\rho}\, \phi_t) \;=\; F_{\mathbf{G}',E',\rho'}\, \phi'_t, \tag{40}$$

where $X' = \psi^{-1}(X)$, $\phi'(X',t) = \phi(X,t)$, $\mathbf{G}' = \psi^*\mathbf{G}$, $E' = \psi^* E = E \circ \psi$, and $\rho' = \psi^*\rho = \rho \circ \psi$.

It is established [M-H] that the equations of elastodynamics are covariant in Eulerian coordinates. Below we extend the argument to Lagrangian coordinates, with special consideration taken for the anisotropy of the energy E in Lagrangian coordinates.

To prove (40) we begin by writing the material acceleration, the first Piola-Kirchhoff stress tensor, and the second Piola-Kirchhoff stress tensor associated with $(\psi^*\mathbf{G}, \psi^*E, \psi^*\rho)$ and ${\phi_t}'$ as $\mathbf{A}'$, $\mathbf{P}'$, and $\mathbf{S}'$, respectively. Note that the representation (31) of $\mathbf{A}$ in terms of the velocity $\mathbf{V}$, which is given in terms of ϕ_t, implies that $\mathbf{A}'(X,t) = \mathbf{A}\big(\psi(X),t\big) = \psi^*\mathbf{A}$. (In particular, $\mathbf{A}$ behaves like a scalar with respect to ψ.) In addition, $\mathbf{S}'^{AB} = 2\rho' \left(\partial(E\circ\psi)/\partial(\psi^*\mathbf{C}^\flat)\right)^{AB}$ since $\mathbf{C}^{\flat\prime} = \phi_t'^*\mathbf{e} = (\phi_t \circ \psi)^*\mathbf{e} = \psi^*\mathbf{C}^\flat$.

Then $\psi^*(F_{\mathbf{G},E,\rho}\,\phi_t) = \rho'\mathbf{A}' - \psi^*(\mathrm{DIV}_{\mathbf{G}}\mathbf{P}) = \rho'\mathbf{A}' - \mathrm{DIV}_{\psi^*\mathbf{G}}\psi^*\mathbf{P}$ by the Piola transform ([M-H], pp. 116–118). We conclude that (40) follows if $\mathbf{P}' = \psi^*\mathbf{P}$.

Now $\mathbf{P}^{aB} = \mathrm{F}^a{}_A S^{AB}$, and $\mathrm{F}'^a{}_C = \partial\phi'^a(X',t)/\partial X'^C = (\mathrm{F}^a{}_A \circ \psi)(D\psi)^A{}_C = (\psi^*\mathbf{F})^a{}_C$, so $(\psi^*\mathbf{P})^{aB'} = (\psi^*\mathbf{F})^a{}_C(\psi^*\mathbf{S})^{CB'} = \mathrm{F}'^a{}_C(\psi^*S)^{CB'}$. Therefore, to show $\psi^*\mathbf{P} = \mathbf{P}'$ we will show $\psi^*\mathbf{S} = \mathbf{S}'$.

Under the constitutive hypothesis for hyperelastic media, the free energy function $E = \overline{E}(X, \mathbf{G}, \mathbf{C}^\flat)$ is related to $S = \bar{S}(X, \mathbf{G}, \mathbf{C}^\flat)$ by

$$S^{AB} \;=\; 2\rho\, \frac{\partial \overline{E}}{\partial {\mathbf{C}^\flat}_{AB}}. \tag{41}$$

(See [M-H], Proposition 4.6.) In addition, the balance laws (conservation of mass, balance of momentum, balance of moment of momentum, and balance of energy) that describe the behavior of hyperelastic media are equivalent to the covariance of the energy [M-H], Theorem 4.13. But the referential energy E is covariant, in the case that E depends functionally just on the point X and the Cauchy-Green deformation tensor $\mathbf{C}$, only if the elastic medium is isotropic.

To include anisotropic media in the discussion we first recall that any particular type of material anisotropy may be characterized by its material symmetry group. (See Section 2.2.) We may associate with any material symmetry group $\mathcal{G}$ a finite collection of *structural tensors* $\boldsymbol{\zeta}_1, \ldots, \boldsymbol{\zeta}_m$ which form a basis for the space of invariant tensors under the action of $\mathcal{G}$; that is, the structural tensors $\boldsymbol{\zeta}_i$ (of order k_i), $i = 1, \ldots, m$, satisfy

$$\mathbf{Q} \in \mathcal{G} \qquad \text{iff} \qquad \langle \mathbf{Q} \rangle_{k_1} \boldsymbol{\zeta}_1 \;=\; \boldsymbol{\zeta}_1, \;\; \ldots, \;\; \langle \mathbf{Q} \rangle_{k_m} \boldsymbol{\zeta}_m \;=\; \boldsymbol{\zeta}_m,$$

where the Kronecker product $\langle \mathbf{Q} \rangle_k$ of an orthogonal transformation $\mathbf{Q}$ is defined by

$$(\langle \mathbf{Q} \rangle_k \zeta)^{i \ldots j} \;=\; Q^i{}_{\underline{i}} \cdots Q^j{}_{\underline{j}} \zeta^{\underline{i} \ldots \underline{j}} \tag{42}$$

for any tensor $\boldsymbol{\zeta}$ of order k, with $Q^i{}_j$, $\zeta^{i \ldots j}$ the Cartesian components of $\mathbf{Q}$ and $\boldsymbol{\zeta}$.

Therefore, we write, given [M-H], p. 220, [L-P], and [Z-Sp], the energy E as a function $E = \widehat{E}(\mathbf{G}, \mathbf{C}^\flat, \boldsymbol{\zeta}_1, \ldots, \boldsymbol{\zeta}_m)$ of $\mathbf{G}$, $\mathbf{C}^\flat$, and the structural tensors $\boldsymbol{\zeta}_1, \ldots, \boldsymbol{\zeta}_m$. It follows from (41) that $S^{AB} = 2\rho \left(\partial \widehat{E} / \partial \mathbf{C}^\flat \right)^{AB} = 2\rho \, \partial \widehat{E} / \partial \mathbf{C}^\flat{}_{AB}$.

Given any diffeomorphism $\psi : \bar{\Omega} \to \bar{\Omega}$, we define the function $\check{E}$ of $\bar{\mathbf{G}}$, $\bar{\mathbf{C}}^\flat, \bar{\boldsymbol{\zeta}}_1, \ldots, \bar{\boldsymbol{\zeta}}_m$ by $\check{E}(\bar{\mathbf{G}}, \bar{\mathbf{C}}^\flat, \bar{\boldsymbol{\zeta}}_1, \ldots, \bar{\boldsymbol{\zeta}}_m) = [\widehat{E}(\psi_* \bar{\mathbf{G}}, \psi_* \bar{\mathbf{C}}^\flat, \psi_* \bar{\boldsymbol{\zeta}}_1, \ldots, \psi_* \bar{\boldsymbol{\zeta}}_m)] \circ \psi$. Then, due to the covariance of the energy, $\psi^* E = E \circ \psi = \check{E}(\psi^* \mathbf{G}, \psi^* \mathbf{C}^\flat, \psi^* \boldsymbol{\zeta}_1, \ldots, \psi^* \boldsymbol{\zeta}_m)$. It follows that

$$\psi^* \Big(\frac{\partial \widehat{E}}{\partial \mathbf{C}^\flat} \Big)^{AB} \;=\; (D\psi^{-1})^A{}_{\underline{A}} (D\psi^{-1})^B{}_{\underline{B}} \cdot \Big(\frac{\partial \widehat{E}}{\partial \mathbf{C}^\flat} \Big)^{\underline{A}\,\underline{B}} \circ \psi \tag{43}$$

and

$$\begin{aligned} \Big(\frac{\partial \widehat{E}}{\partial \mathbf{C}^\flat} \Big)^{\underline{A}\,\underline{B}} \circ \psi \;&=\; \Big(\frac{\partial \check{E}}{\partial (\psi^* \mathbf{C}^\flat)} \Big)^{A'B'} \cdot \frac{\partial (\psi^* \mathbf{C}^\flat \circ \psi^{-1})_{A'B'}}{\partial \mathbf{C}^\flat_{\underline{A}\,\underline{B}}} \\ &=\; \Big(\frac{\partial \check{E}}{\partial (\psi^* \mathbf{C}^\flat)} \Big)^{A'B'} \cdot (D\psi)^{\underline{A}}{}_{A'} (D\psi)^{\underline{B}}{}_{B'}. \end{aligned} \tag{44}$$

Consequently, $\psi^* \mathbf{S} = \mathbf{S}'$ since

$$\psi^* \mathbf{S} \;=\; 2(\rho \circ \psi)\, \psi^* \big(\frac{\partial \widehat{E}}{\partial \mathbf{C}^\flat} \big) \;=\; 2\rho' \, \frac{\partial \check{E}}{\partial (\psi^* \mathbf{C}^\flat)} \;=\; \mathbf{S}'. \tag{45}$$

The Dirichlet-to-Neumann maps agree in this case because $\psi \in \Psi$ is the identity at the boundary to first order, which implies that $\mathbf{G}' = \psi^* \mathbf{G}$ is $\mathbf{G}$ at $\partial\Omega$ and $\mathbf{P}' = \psi^* \mathbf{P}$ is $\mathbf{P}$ at $\partial\Omega$. That is, $\Lambda_{\mathbf{G}, E, \rho} f \;=\; \langle \mathbf{P}, \boldsymbol{\nu} \rangle \sqrt{\det \mathbf{G}} \; dS$ agrees with $\Lambda_{\mathbf{G}', E', \rho'} f \;=\; \langle \mathbf{P}', \boldsymbol{\nu} \rangle \sqrt{\det \mathbf{G}'} \; dS$ at $\partial\Omega$.

The linearized case. In this case we model the displacement of the hyperelastic body by the initial-boundary-value problem (35). To check that the equations of motion (35a) are covariant, we consider a motion $\mathring{\phi}_t : (\Omega, \mathbf{G}) \to (\mathcal{S}, \mathbf{g})$ of an elastic body $(\Omega, \mathbf{G}, \mathring{E}, \mathring{\rho})$. Given a diffeomorphism $\psi \in \Psi$, we define $\mathring{\phi}'(X', t) : (\Omega, \psi^* \mathbf{G}) \to (\mathcal{S}, \mathbf{g})$ by $\mathring{\phi}'(X', t) = \mathring{\phi}(\psi(X'), t)$, where $X' = \psi^{-1}(X)$. It follows from the proof of covariance for the non-linear case that $\mathring{\phi}'$ is a motion of the transformed elastic body $(\Omega, \mathring{\mathbf{G}}', \mathring{E}', \mathring{\rho}')$, where $\mathring{\mathbf{G}}' = \psi^* \mathring{\mathbf{G}}$, $\mathring{\rho}' = \mathring{\rho} \circ \psi$, and $\mathring{E}' = \mathring{E} \circ \psi$. The same proof shows that

$\mathring{\mathbf{F}}' = \psi^*\mathring{\mathbf{F}}$, $\mathring{\mathbf{P}}' = \psi^*\mathring{\mathbf{P}}$, and $\mathring{\mathbf{S}}' = \psi^*\mathring{\mathbf{S}}$. Also, $\mathring{\mathbf{C}}' = \psi^*\mathring{\mathbf{C}}$ by (29) and the reasoning in (43) and (45).

It follows that $\mathring{\mathbf{A}}' = \psi^*\mathring{\mathbf{A}}$ since $\mathbf{A}$ depends tensorially on $\mathbf{C}$, $\mathbf{F}$, and the metric. Explicitly, in components:

(46)
$$\begin{aligned}
\mathring{\mathsf{A}}'^{aA'}{}_b{}^{B'} &= 2\,\mathring{\mathsf{C}}'^{E'A'F'B'}\mathring{\mathrm{F}}'^a{}_{E'}\mathring{\mathrm{F}}'^c{}_{F'}(\mathring{g}_{bc}\circ\phi') \;+\; \mathring{\mathbf{S}}'^{A'B'}(\delta^a{}_b\circ\phi') \\
&= 2(D\psi^{-1})^{E'}{}_E(D\psi^{-1})^{A'}{}_A(D\psi^{-1})^{F'}{}_F(D\psi^{-1})^{B'}{}_B(\mathring{\mathsf{C}}^{EAFB}\circ\psi) \\
&\quad\cdot\ (\mathring{\mathrm{F}}^a{}_E\mathring{\mathrm{F}}^c{}_F)\circ\psi(D\psi)^E{}_{E'}(D\psi)^F{}_{F'}(\mathring{g}_{bc}\circ\phi)\circ\psi \\
&\quad+\ (D\psi^{-1})^{A'}{}_A(D\psi^{-1})^{B'}{}_B(\mathring{\mathbf{S}}^{AB}\circ\psi)\ (\delta^a{}_b\circ\phi)\circ\psi \\
&= (D\psi^{-1})^{A'}{}_A(D\psi^{-1})^{B'}{}_B\ (\mathring{\mathsf{A}}^{aA}{}_b{}^B\circ\psi).
\end{aligned}$$

We conclude that

$$\begin{aligned}
[\nabla_{\psi^*\mathbf{G}}(\mathbf{U}\circ\psi)]^b{}_{B'} &= \frac{\partial(U^b\circ\psi)}{\partial X'^{B'}} \;+\; (\gamma^b_{\underline{a}c}\circ\phi')(U^{\underline{a}}\circ\psi)\mathrm{F}'^c{}_{B'} \\
&= [(\nabla_{\mathbf{G}}\mathbf{U})^b{}_B\circ\psi]\ (D\psi)^B{}_{B'},
\end{aligned} \tag{47}$$

where $[\nabla_{\mathbf{G}}\mathbf{U}]^b{}_B = \partial\mathbf{U}^b/\partial X^B + (\gamma^b_{ac}\circ\phi)\,\mathbf{U}^a\,\mathrm{F}^c{}_B$ is the covariant derivative of the two-point vector $\mathbf{U}$. Therefore, the linearized stress $[\mathring{\mathbf{A}}\cdot\nabla_{\mathbf{G}}\mathbf{U}]^{aA} = \mathring{\mathsf{A}}^{aA}{}_b{}^B[\nabla_{\mathbf{G}}\mathbf{U}]^b{}_B$ satisfies $[\mathring{\mathbf{A}}'\cdot\nabla_{\mathbf{G}'}\mathbf{U}']^{aA'} = \mathring{\mathsf{A}}'^{aA'}{}_b{}^{B'}[\nabla_{\psi^*\mathbf{G}}(\mathbf{U}\circ\psi)]^b{}_{B'} = (D\psi^{-1})^{A'}{}_A([\mathring{\mathbf{A}}\cdot\nabla_{\mathbf{G}}\mathbf{U}]^{aA}\circ\psi) = (\psi^*[\mathring{\mathbf{A}}\cdot\nabla_{\mathbf{G}}U])^{aA'}$, where $\mathbf{U}' = \mathbf{U}\circ\psi$.

For completeness we check below that $(\mathrm{DIV}_{X'}[\mathring{\mathbf{A}}'\cdot\nabla_{\mathbf{G}'}\mathbf{U}'])^a = (\mathrm{DIV}_X[\mathring{\mathbf{A}}\cdot\nabla_{\mathbf{G}}\mathbf{U}])^a\circ\psi$. (See (32) for the definition of the divergence of a two-point tensor.) We first define $\epsilon_{ABC} = 1$ if A, B, C is an even permutation of $1,2,3$, $\epsilon_{ABC} = -1$ if A, B, C is an odd permutation of $1,2,3$, and $\epsilon_{ABC} = 0$ otherwise. Then the cofactor matrix Cof M of a 3×3 matrix M is given by $(\,\mathrm{Cof}\ M)^{A'}{}_A = (1/2)\epsilon_{ABC}\ \epsilon^{A'B'C'}\ M^B{}_{B'}\,M^C{}_{C'}$. It follows that $\partial/\partial X'^{A'}(\,\mathrm{Cof}\ D\psi)^{A'}{}_A = \sum_{B,C,C'}\epsilon_{ABC}\ \cdot\ 0\ \cdot\partial X^C/\partial X'^{C'} = 0$. Also, $\sum_A\Gamma^A_{AB} = \partial/\partial X^B(\log\sqrt{\det\ \mathbf{G}})$.

Writing $M^{-1} = \mathrm{Cof}\ M/\det\ M$ and $\partial/\partial X'^{A'} = (D\psi)^A{}_{A'}(\partial/\partial X^A)$, we may therefore conclude that covariance holds, since

$$\begin{aligned}
&(\mathrm{DIV}_{X'}[\mathring{\mathbf{A}}'\cdot\nabla_{\mathbf{G}'}\mathbf{U}'])^a \\
&\quad= \frac{\partial}{\partial X'^{A'}}[\mathring{\mathbf{A}}'\cdot\nabla_{\mathbf{G}'}\mathbf{U}']^{aA'} \;+\; \Gamma'^{\underline{A}}_{\underline{A}A'}\,[\mathring{\mathbf{A}}'\cdot\nabla_{\mathbf{G}'}\mathbf{U}']^{aA'} \\
&\qquad\qquad+\ (\gamma^a_{\underline{b}\,\underline{a}}\circ\phi')\mathring{\mathrm{F}}'^{\underline{b}}{}_{A'}[\mathring{\mathbf{A}}'\cdot\nabla_{\mathbf{G}'}\mathbf{U}']^{\underline{a}A'}
\end{aligned} \tag{48}$$

$$
\begin{aligned}
&= \frac{\partial}{\partial X'^{A'}}\left[(D\psi^{-1})^{A'}{}_{A}([\mathring{\mathbf{A}}\cdot\nabla_{\mathbf{G}}\mathbf{U}]^{aA}\circ\psi)\right]\\
&\quad + \Big(\frac{\partial}{\partial X'^{A'}}\log\sqrt{\det\,\psi^*\mathbf{G}}\Big)(D\psi^{-1})^{A'}{}_{A}([\mathring{\mathbf{A}}\cdot\nabla_{\mathbf{G}}\mathbf{U}]^{aA}\circ\psi)\\
&\quad + \big((\gamma^{a}_{\underline{b}\,\underline{a}}\circ\phi)\circ\psi\big)\,(\mathring{\mathrm{F}}^{\underline{b}}{}_{A}\circ\psi)\,([\mathring{\mathbf{A}}\cdot\nabla_{\mathbf{G}}\mathbf{U}]^{\underline{a}A}\circ\psi)\\
&= \frac{\partial}{\partial X'^{A'}}\left[(\det\,D\psi^{-1})\,\mathrm{Cof}\,(D\psi)^{A'}{}_{A}\;([\mathring{\mathbf{A}}\cdot\nabla_{\mathbf{G}}\mathbf{U}]^{aA}\circ\psi)\right]\\
&\quad + \Big(\frac{\partial}{\partial X'^{A'}}\log(\det\,D\psi)\;+\;(D\psi)^{A}{}_{A'}\Gamma^{\underline{A}}_{\underline{A}A}\Big)\\
&\qquad\qquad\cdot(D\psi^{-1})^{A'}{}_{A}([\mathring{\mathbf{A}}\cdot\nabla_{\mathbf{G}}\mathbf{U}]^{aA}\circ\psi)\\
&\quad + \Big((\gamma^{a}_{\underline{b}\,\underline{a}}\circ\phi)\,\mathring{\mathrm{F}}^{\underline{b}}{}_{A}\,[\mathring{\mathbf{A}}\cdot\nabla_{\mathbf{G}}\mathbf{U}]^{\underline{a}A}\Big)\circ\psi\\
&= (\mathrm{DIV}_X[\mathring{\mathbf{A}}\cdot\nabla_{\mathbf{G}}\mathbf{U}])^{a}\circ\psi.
\end{aligned}
\tag{49}
$$

It follows, since $\psi^*(\mathrm{DIV}_{\mathbf{G}}\mathring{\mathbf{P}}) = \mathrm{DIV}_{\psi^*\mathbf{G}}\psi^*\mathring{\mathbf{P}}$, that

$$
\begin{aligned}
&\psi^*(P_{\mathbf{G},\mathring{\mathrm{A}},\mathring{E},\mathring{\rho},\mathring{\mathbf{S}},\mathring{\mathbf{P}}}\,\mathbf{U})\\
&= (\mathring{\rho}\circ\psi)\Big(\frac{\partial^2\mathbf{U}(X,t)\circ\psi}{\partial t^2}+\psi^*\mathring{\mathbf{A}}\Big)-\psi^*\Big(\mathrm{DIV}_{\mathbf{G}}(\mathring{\mathbf{A}}\cdot\nabla_{\mathbf{G}}\mathbf{U}+\mathring{\mathbf{P}})\Big)\\
&= P_{\mathbf{G}',\mathring{\mathrm{A}}',\mathring{E}',\mathring{\rho}',\mathring{\mathbf{S}}',\mathring{\mathbf{P}}'}\,\mathbf{U}'.
\end{aligned}
\tag{50}
$$

Again, the Dirichlet-to-Neumann maps agree because $\psi\in\Psi$ is the identity at the boundary, which implies that $\mathbf{G}'=\psi^*\mathbf{G}$ is $\mathbf{G}$ at $\partial\Omega$, $\mathbf{P}'=\psi^*\mathbf{P}$ is $\mathbf{P}$ at $\partial\Omega$, and $\mathring{\mathbf{A}}'\cdot\nabla_{\mathbf{G}'}\mathbf{U}'=\psi^*[\mathring{\mathbf{A}}\cdot\nabla_{\mathbf{G}}U]$ is $\mathring{\mathbf{A}}\cdot\nabla_{\mathbf{G}}\mathbf{U}$ at $\partial\Omega$. That is, $\Lambda_{\mathbf{G}',\mathring{\mathrm{A}}',\mathring{E}',\mathring{\rho}',\mathring{\mathbf{S}}',\mathring{\mathbf{P}}'}\,f=\langle[\mathring{\mathbf{A}}'\cdot\nabla_{\mathbf{G}'}\mathbf{U}']+\mathring{\mathbf{P}}',\boldsymbol{\nu}\rangle\sqrt{\det\,\mathbf{G}'}\,dS$ agrees with $\Lambda_{\mathbf{G},\mathring{\mathrm{A}},\mathring{E},\mathring{\rho},\mathring{\mathbf{S}},\mathring{\mathbf{P}}}\;f=\langle[\mathring{\mathbf{A}}\cdot\nabla_{\mathbf{G}}\mathbf{U}]+\mathring{\mathbf{P}},\boldsymbol{\nu}\rangle\sqrt{\det\,\mathbf{G}}\;dS$ where $\langle\mathbf{P},\boldsymbol{\nu}\rangle^a=\mathbf{P}^{aA}\delta_{AB}\boldsymbol{\nu}^B$.

The classical linear case. Here we take $\mathring{\phi}_t$ to be a stress-free, undeformed state $\mathring{\phi}_t(X,t)=\iota(X)$, where ι is an embedding $(\Omega,\mathbf{G})\to(\mathcal{S},\mathbf{g})$. Therefore, $\mathring{\mathbf{S}}$, $\mathring{\mathbf{P}}=0$, and $\mathring{\mathbf{A}}=0$. We observe that $\mathring{\phi}'(X,t)=\mathring{\phi}(\psi(X),t)$ is also stress-free since the stress tensor $\mathbf{S}$ associated with a motion is covariant, so $\mathring{\mathbf{S}}=0$ implies $\mathring{\mathbf{S}}'=\psi^*\mathring{\mathbf{S}}=0$.

In applications, we take $\mathbf{g}=\mathbf{e}$ so that $\gamma^a_{bc}=0$. Then, a calculation similar to (48) along with (37) establishes the local-coordinate representation (6) of the equations of motion. □

Remark 16. For classical linear elasticity, a weak formulation of the equations of motion, in terms of Dirichlet integrals, holds (see [M-H]). In this case, as it happens with the scalar wave equation, it may be possible to use the weak formulation to obtain obstruction to uniqueness for diffeomorphisms ψ that fix the boundary to zeroth-order only; i.e., for which $D\psi$ is not necessarily the identity at the boundary.

5. Proof of Theorem 11

5.1. The harmonic-Cartan decomposition of $\bar{\mathsf{C}}$. We write $\bar{\mathsf{C}}$ (at any point $X \in \Omega$ and with respect to any orthogonal basis e_1, e_2, e_3, here the standard basis for Euclidean space) in terms of its harmonic decomposition (as described in [F-V])

$$\bar{\mathsf{C}} \;=\; \bar{\alpha} I \otimes I \;+\; \bar{\beta} I \diamondsuit I \;+\; 2 I \circledS \bar{A} \;+\; 2 I \diamondsuit \bar{B} \;+\; \bar{\mathsf{H}}, \tag{51}$$

where $\bar{\alpha}$ and $\bar{\beta}$ are scalars, $\bar{A}$ and $\bar{B}$ are symmetric, traceless 3×3 matrices (i.e. with trace $\bar{A}^{11} + \bar{A}^{22} + \bar{A}^{33}$ equal to zero), and $\bar{\mathsf{H}}$ is a totally symmetric rank-four tensor (i.e. $\bar{\mathsf{H}}^{ijkl} = \bar{\mathsf{H}}^{\sigma(ijkl)}$, where σ is any permutation of the (i, j, k, l) indices). In addition, $\bar{\mathsf{H}}$ is harmonic; that is, for $\mathbf{r} = xe_1 + ye_2 + ze_3$,
(52)

$$\bar{\mathsf{H}}[\mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{r}] = \sum_{i,j,k,l=1}^{3} \bar{\mathsf{H}}^{ijkl} x^{\delta_{i1}+\delta_{j1}+\delta_{k1}+\delta_{l1}} y^{\delta_{i2}+\delta_{j2}+\delta_{k2}+\delta_{l2}} z^{\delta_{i3}+\delta_{j3}+\delta_{k3}+\delta_{l3}}$$

is a harmonic polynomial in x, y, z, i.e. $(\partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2)\bar{\mathsf{H}}[\mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{r}] = 0$. Here δ_{ij} is the Kronecker delta, and we use the notation

$$\begin{aligned}
(A \circledS B)^{ijkl} &= \frac{1}{2}(A \otimes B \;+\; B \otimes A)^{ijkl} \;=\; \frac{1}{2}(A^{ij}B^{kl} \;+\; B^{ij}A^{kl}) \\
A \diamondsuit B &= \frac{1}{2}(A \diamondsuit B \;+\; B \diamondsuit A) \\
(A \diamondsuit B)^{ijkl} &= A^{ik}B^{jl} \;+\; A^{il}B^{jk}.
\end{aligned} \tag{53}$$

The harmonic decomposition (51) of $\bar{\mathsf{C}}$ may be written in terms of the components of $\bar{\mathsf{C}}$ as follows:

$$\begin{aligned}
\bar{\alpha} &= \frac{1}{15}\big(2\bar{\mathsf{C}}^{ppkk} \;-\; \bar{\mathsf{C}}^{pkpk}\big) \\
\bar{\beta} &= \frac{1}{30}\big(-\bar{\mathsf{C}}^{ppkk} \;+\; 3\bar{\mathsf{C}}^{pkpk}\big) \\
\bar{A}^{ij} &= \frac{1}{21}\big(15\bar{\mathsf{C}}^{ijkk} \;-\; 12\bar{\mathsf{C}}^{ikjk} \;-\; 5\delta^{ij}\bar{\mathsf{C}}^{ppkk} \;+\; 4\delta^{ij}\bar{\mathsf{C}}^{pkpk}\big) \\
\bar{B}^{ij} &= \frac{-1}{21}\big(6\bar{\mathsf{C}}^{ijkk} \;-\; 9\bar{\mathsf{C}}^{ikjk} \;-\; 2\delta^{ij}\bar{\mathsf{C}}^{ppkk} \;+\; 3\delta^{ij}\bar{\mathsf{C}}^{pkpk}\big) \\
\bar{H}^{ijkl} &= \frac{1}{3}\big(\bar{\mathsf{C}}^{ijkl} \;+\; \bar{\mathsf{C}}^{ikjl} \;+\; \bar{\mathsf{C}}^{iljk}\big) \\
&\quad - \frac{1}{21}\Big[(\bar{\mathsf{C}}^{ijmm} \;+\; 2\bar{\mathsf{C}}^{imjm})\delta^{kl} \;+\; (\bar{\mathsf{C}}^{klmm} \;+\; 2\bar{\mathsf{C}}^{kmlm})\delta^{ij} \\
&\qquad + (\bar{\mathsf{C}}^{ikmm} \;+\; 2\bar{\mathsf{C}}^{imkm})\delta^{jl} \;+\; (\bar{\mathsf{C}}^{jlmm} \;+\; 2\bar{\mathsf{C}}^{imjm})\delta^{ik} \\
&\qquad + (\bar{\mathsf{C}}^{ilmm} \;+\; 2\bar{\mathsf{C}}^{imlm})\delta^{jk} \;+\; (\bar{\mathsf{C}}^{jkmm} \;+\; 2\bar{\mathsf{C}}^{imjm})\delta^{il}\Big] \;+
\end{aligned} \tag{54}$$

$$+\frac{1}{105}\big(\bar{\mathsf{C}}^{ppmm}+2\bar{\mathsf{C}}^{pmpm}\big)\big(\delta^{ij}\delta^{kl}\ +\ \delta^{ik}\delta^{jl}\ +\ \delta^{il}\delta^{jk}\big). \tag{55}$$

(Note the typographical error in [F-V] in the expression for $\bar{\beta}$.)

We then write $\bar{\mathsf{H}}$ in terms of the Cartan decomposition (as described in [F-V])

$$\bar{\mathsf{H}}\ =\ \bar{\alpha}_0\mathbf{U}\ +\ \sum_{m=1}^{4}\bar{\alpha}_m\boldsymbol{\mathcal{S}}_m\ +\ \bar{\beta}_m\boldsymbol{\mathcal{T}}_m, \tag{56}$$

where $\bar{\alpha}_0,\bar{\alpha}_i,\bar{\beta}_i,\ i\ =\ 1...4,$ are scalars, and the harmonic tensors $\mathbf{U},\boldsymbol{\mathcal{S}}_i,\boldsymbol{\mathcal{T}}_i,\ i=1...4,$ have polynomial form

$$\begin{aligned}
\mathbf{U}[\mathbf{r},\mathbf{r},\mathbf{r},\mathbf{r}] &= u(x,y,z) = 8z^4+3(x^2+y^2)(x^2+y^2-8z^2)\\
\boldsymbol{\mathcal{S}}_1[\mathbf{r},\mathbf{r},\mathbf{r},\mathbf{r}] &= s_1(x,y,z) = zx\big(4z^2-3[x^2+y^2]\big)\\
\boldsymbol{\mathcal{T}}_1[\mathbf{r},\mathbf{r},\mathbf{r},\mathbf{r}] &= t_1(x,y,z) = zy\big(4z^2-3[x^2+y^2]\big)\\
\boldsymbol{\mathcal{S}}_2[\mathbf{r},\mathbf{r},\mathbf{r},\mathbf{r}] &= s_2(x,y,z) = (x^2-y^2)\big(6z^2-[x^2+y^2]\big)\\
\boldsymbol{\mathcal{T}}_2[\mathbf{r},\mathbf{r},\mathbf{r},\mathbf{r}] &= t_2(x,y,z) = 2xy\big(6z^2-[x^2+y^2]\big)\\
\boldsymbol{\mathcal{S}}_3[\mathbf{r},\mathbf{r},\mathbf{r},\mathbf{r}] &= s_3(x,y,z) = zx(x^2-3y^2)\\
\boldsymbol{\mathcal{T}}_3[\mathbf{r},\mathbf{r},\mathbf{r},\mathbf{r}] &= t_3(x,y,z) = zy(3x^2-y^2)\\
\boldsymbol{\mathcal{S}}_4[\mathbf{r},\mathbf{r},\mathbf{r},\mathbf{r}] &= s_4(x,y,z) = x^4+y^4-6x^2y^2\\
\boldsymbol{\mathcal{T}}_4[\mathbf{r},\mathbf{r},\mathbf{r},\mathbf{r}] &= t_4(x,y,z) = 4xy(x^2-y^2).
\end{aligned} \tag{57}$$

We refer to their polynomial form (52) to compute the components of U, for example:

$$\begin{gathered}
U^{3,3,3,3}=8,\quad U^{1,1,1,1}=3,\quad U^{2,2,2,2}=3,\quad \sum_{\sigma\in S_3}U^{\sigma(1,1,3,3)}=-24,\\
\sum_{\sigma\in S_3}U^{\sigma(2,2,3,3)}=-24,\quad \sum_{\sigma\in S_3}U^{\sigma(1,1,2,2)}=6,
\end{gathered} \tag{58}$$

where S_3 is the symmetric group (of permutations of three elements).

It follows that the components $\bar{\mathsf{H}}^{ijkl}$ of $\bar{\mathsf{H}}$ are given by the following:

(59)

$ij\backslash kl$	11	22	33
11	$3\bar{\alpha}_0-\bar{\alpha}_2+\bar{\alpha}_4$	$\bar{\alpha}_0-\bar{\alpha}_4$	$-4\bar{\alpha}_0+\bar{\alpha}_2$
22		$3\bar{\alpha}_0+\bar{\alpha}_2+\bar{\alpha}_4$	$-4\bar{\alpha}_0-\bar{\alpha}_2$
33			$8\bar{\alpha}_0$

(60)

$ij\backslash kl$	12	23	13
11	$-\bar{\beta}_2/2+\bar{\beta}_4$	$-[\bar{\beta}_1-\bar{\beta}_3]/4$	$-[3\bar{\alpha}_1-\bar{\alpha}_3]/4$
22	$-\bar{\beta}_2/2-\bar{\beta}_4$	$-[3\bar{\beta}_1+\bar{\beta}_3]/4$	$-[\bar{\alpha}_1+\bar{\alpha}_3]/4$
33	$\bar{\beta}_2$	$\bar{\beta}_1$	$\bar{\alpha}_1$
12	$\bar{\alpha}_0-\bar{\alpha}_4$	$-[\bar{\alpha}_1+\bar{\alpha}_3]/4$	$-[\bar{\beta}_1-\bar{\beta}_3]/4$
23		$-4\bar{\alpha}_0-\bar{\alpha}_2$	$\bar{\beta}_2$
13			$-4\bar{\alpha}_0+\bar{\alpha}_2$

while, writing $M=(\bar{\alpha}+2\bar{\beta})I+2(\bar{A}+2\bar{B})$, $\bar{\mathbf{C}}-\bar{\mathbf{H}}$ is given by:

(61)

$ij\backslash kl$	11	22	33
11	M^{11}	$\bar{\alpha}+\bar{A}^{11}+\bar{A}^{22}$	$\bar{\alpha}-\bar{A}^{22}$
22		M^{22}	$\bar{\alpha}-\bar{A}^{11}$
33			M^{33}

(62)

$ij\backslash kl$	12	23	13
11	$(\bar{A}+2\bar{B})^{12}$	$\bar{A}^{23}$	$(\bar{A}+2\bar{B})^{13}$
22	$(\bar{A}+2\bar{B})^{12}$	$(\bar{A}+2\bar{B})^{23}$	$\bar{A}^{13}$
33	$\bar{A}^{12}$	$(\bar{A}+2\bar{B})^{23}$	$(\bar{A}+2\bar{B})^{13}$
12	$\bar{\beta}+\bar{B}^{11}+\bar{B}^{22}$	$\bar{B}^{13}$	$\bar{B}^{23}$
23		$\bar{\beta}-\bar{B}^{11}$	$\bar{B}^{12}$
13			$\bar{\beta}-\bar{B}^{22}$

5.2. **Describing $\mathrm{Sym}_e\bar{\mathbf{C}}$ in terms of $\bar{A},\bar{B},\bar{\alpha}_i,\bar{\beta}_i$.** The symmetry group of $\bar{\mathbf{C}}$ is given [F-V] in terms of the symmetry groups of $\bar{A}$, $\bar{B}$, and $\bar{\mathbf{H}}$ by

$$\mathrm{Sym}_e(\bar{\mathbf{C}}) \;=\; \mathrm{Sym}_e(\bar{A}) \;\cap\; \mathrm{Sym}_e(\bar{B}) \;\cap\; \mathrm{Sym}_e(\bar{\mathbf{H}}). \tag{63}$$

In addition, following the reasoning given in [F-V] we present in Table 2 a more precise description of $\mathrm{Sym}_e\bar{\mathbf{H}}$, almost entirely in terms of the vanishing of $\bar{\alpha}_0,\bar{\alpha}_i,\bar{\beta}_i$, $i=1..4$. Note that each of the entries (i.e. sets of rows) of Table 2 is numbered as in Table 1 to indicate its type of (an)isotropy. In Table 2 and below we denote by $\mathcal{Z}_n,\mathcal{D}_n,\mathcal{O},SO(n),O(n)$ the cyclic, dihedral, octahedral, special orthogonal, and orthogonal groups. We emphasize that the entries in Table 2 depend on a choice of orthogonal basis $\{e_1,e_2,e_3\}$ since the $\bar{\alpha}_0,\bar{\alpha}_i,\bar{\beta}_i$ may change with a change of basis.

We observe that the polynomial forms $\mathbf{r}^T\bar{A}\mathbf{r}=\bar{A}_{11}x^2+2\bar{A}_{12}xy+\dots$ and $\mathbf{r}^T\bar{B}\mathbf{r}$ of the harmonic 2-tensors $\bar{A}$ and $\bar{B}$ may also be decomposed in terms of the basis

$$\tilde{u} \;=\; z^2-\frac{1}{2}(x^2+y^2), \qquad \tilde{s}_1 \;=\; zx, \qquad \tilde{t}_1 \;=\; zy,$$
$$\tilde{s}_2 \;=\; x^2-y^2, \qquad \tilde{t}_2 \;=\; 2xy$$

TABLE 2. $\mathrm{Sym}_e\bar{\mathbf{H}}$ **is described by the vanishing of** $\bar\alpha_0, \bar\alpha_i, \bar\beta_i$

	Properties of $\bar\alpha_0, \bar\alpha_i, \bar\beta_i$	Restrictions on $\mathrm{Sym}_e\bar{\mathbf{H}}$
1	$\bar\alpha_1 \neq 0$ or $\bar\alpha_3 \neq 0$	$\mathcal{Z}_2 \cong \mathrm{sg}\{Q^\pi_{e_1}\},\ \mathrm{sg}\{Q^\pi_{e_3}\} \not\subseteq \mathrm{Sym}_e\bar{\mathbf{H}}$
1	$\bar\beta_1 \neq 0$ or $\bar\beta_3 \neq 0$	$\mathcal{Z}_2 \cong \mathrm{sg}\{Q^\pi_{e_2}\},\ \mathrm{sg}\{Q^\pi_{e_3}\} \not\subseteq \mathrm{Sym}_e\bar{\mathbf{H}}$
1	$\bar\beta_2 \neq 0$ or $\bar\beta_4 \neq 0$	$\mathcal{Z}_2 \cong \mathrm{sg}\{Q^\pi_{e_1}\},\ \mathrm{sg}\{Q^\pi_{e_2}\} \not\subseteq \mathrm{Sym}_e\bar{\mathbf{H}}$
2	$\bar\alpha_1, \bar\beta_1, \bar\alpha_3, \bar\beta_3 = 0$ with $\bar\beta_2 \neq 0$ or $\bar\beta_4 \neq 0$ with $\bar\alpha_2 \neq 0$	$\mathcal{Z}_2 \cong \mathrm{sg}\{Q^\pi_{e_3}\} \subseteq \mathrm{Sym}_e\bar{\mathbf{H}}$ $\mathcal{D}_2$, $\mathrm{sg}\{Q^\pi_{e_1}\}$, $\mathrm{sg}\{Q^\pi_{e_2}\} \not\subseteq \mathrm{Sym}_e\bar{\mathbf{H}}$ $\mathcal{Z}_3 \cong \mathrm{sg}\{Q^{2\pi/3}_{e_i}\} \not\subseteq \mathrm{Sym}_e\bar{\mathbf{H}}$ $\mathcal{Z}_4 \cong \mathrm{sg}\{Q^{\pi/2}_{e_i}\} \not\subseteq \mathrm{Sym}_e\bar{\mathbf{H}}$
3	$\bar\alpha_1, \bar\beta_1, \bar\beta_2, \bar\alpha_3, \bar\beta_3, \bar\beta_4 = 0$ with $\bar\alpha_2 \neq 0$	$\mathcal{D}_2 \cong \mathrm{sg}\{Q^\pi_{e_1}, Q^\pi_{e_2}, Q^\pi_{e_3}\} \subseteq \mathrm{Sym}_e\bar{\mathbf{H}}$ $\mathcal{Z}_3 \cong \mathrm{sg}\{Q^{2\pi/3}_{e_i}\} \not\subseteq \mathrm{Sym}_e\bar{\mathbf{H}}$ $\mathcal{Z}_4 \cong \mathrm{sg}\{Q^{\pi/2}_{e_i}\} \not\subseteq \mathrm{Sym}_e\bar{\mathbf{H}}$
4	$\bar\alpha_1, \bar\beta_1, \bar\alpha_2, \bar\beta_2, \bar\alpha_3, \bar\alpha_4, \bar\beta_4 = 0$ with $\bar\beta_3 \neq 0$	$\mathcal{D}_3 \cong \mathrm{sg}\{Q^\pi_{e_1}, Q^{2\pi/3}_{e_3}\} \subseteq \mathrm{Sym}_e\bar{\mathbf{H}}$ $O(2)$, $\mathrm{sg}\{Q^{2\pi/3}_{e_{j\neq 3}}\}$, $\mathrm{sg}\{Q^\pi_{e_3}\} \not\subseteq \mathrm{Sym}_e\bar{\mathbf{H}}$
5	$\bar\alpha_1, \bar\beta_1, \bar\alpha_2, \bar\beta_2, \bar\alpha_3, \bar\beta_3, \bar\beta_4 = 0$ with $\bar\alpha_4 \neq 0$	$\mathcal{D}_4 \cong \mathrm{sg}\{Q^\pi_{e_1}, Q^\pi_{e_2}, Q^{\pi/2}_{e_3}\} \subseteq \mathrm{Sym}_e\bar{\mathbf{H}}$ $\mathcal{O}$, $\mathrm{sg}\{Q^{2\pi/3}_{e_i}\}$, $\mathrm{sg}\{Q^{\pi/2}_{e_{j\neq 3}}\} \not\subseteq \mathrm{Sym}_e\bar{\mathbf{H}}$
6	$\bar\alpha_1, \bar\beta_1, \bar\alpha_2, \bar\beta_2, \bar\alpha_3, \bar\beta_3, \bar\beta_4 = 0$ with $\bar\alpha_4 = 5\bar\alpha_0 \neq 0$	$\mathcal{O} \cong \mathrm{sg}\{Q^{\pi/2}_{e_1}, Q^{\pi/2}_{e_2}, Q^{\pi/2}_{e_3}\} \subseteq \mathrm{Sym}_e\bar{\mathbf{H}}$ $O(2) \not\subseteq \mathrm{Sym}_e\bar{\mathbf{H}}$
7	$\bar\alpha_i, \bar\beta_i = 0,\ i = 1...4$ with $\bar\alpha_0 \neq 0$	$O(2) \cong \mathrm{sg}\{Q^\pi_{e_1}, Q^\pi_{e_2}, Q^\phi_{e_3}\} \subseteq \mathrm{Sym}_e\bar{\mathbf{H}}$ $O(3) \not\subseteq \mathrm{Sym}_e\bar{\mathbf{H}}$
8	$\bar\alpha_0, \bar\alpha_i, \bar\beta_i = 0,\ i = 1...4$	$O(3) \subseteq \mathrm{Sym}_e\bar{\mathbf{H}}$

of homogeneous, degree-2 polynomials in x, y, z. In fact,

$$\begin{aligned}\mathbf{r}^T A'\mathbf{r} &= \tilde\alpha_0\tilde u + \tilde\alpha_1\tilde s_1 + \tilde\beta_1\tilde t_1 + \tilde\alpha_2\tilde s_2 + \tilde\beta_2\tilde t_2 \\ &= A'_{33}\tilde u + 2A'_{13}\tilde s_1 + 2A'_{23}\tilde t_1 + \frac{1}{2}(A'_{11} - A'_{22})\tilde s_2 + A'_{12}\tilde t_2\end{aligned}$$

for any symmetric, traceless, rank-2 tensor A'. It follows that the symmetry groups of $\bar A$ and $\bar B$ may be described in terms of the vanishing of their components, as summarized in Table 3.

In addition, as shown in [F-V],

$$\mathrm{Sym}_e(\bar A) \cap \mathrm{Sym}_e(\bar B) \cong \{I\},\ \mathcal{Z}_2,\ \mathcal{D}_2,\ O(2),\ \text{or } SO(3). \tag{64}$$

TABLE 3. $\mathrm{Sym}_e A'$ **is described by the vanishing of its components**

	Properties of A'_{ij}	Restrictions on $\mathrm{Sym}_e A'$
1	$A'_{13} \neq 0$ or $A'_{12} \neq 0$	$\mathcal{Z}_2 \cong \mathrm{sg}\{Q^{\pi}_{e_1}\} \not\subseteq \mathrm{Sym}_e A'$
1	$A'_{23} \neq 0$ or $A'_{12} \neq 0$	$\mathcal{Z}_2 \cong \mathrm{sg}\{Q^{\pi}_{e_2}\} \not\subseteq \mathrm{Sym}_e A'$
1	$A'_{13} \neq 0$ or $A'_{23} \neq 0$	$\mathcal{Z}_2 \cong \mathrm{sg}\{Q^{\pi}_{e_3}\} \not\subseteq \mathrm{Sym}_e A'$
3	$A'_{13}, A'_{23}, A'_{12} = 0$ with $A'_{11}-A'_{22} \neq 0$	$D(2) \cong \mathrm{sg}\{Q^{\pi}_{e_1}, Q^{\pi}_{e_2}, Q^{\pi}_{e_3}\} \subseteq \mathrm{Sym}_e A'$ $\mathrm{sg}\{Q^{\pi/2}_{e_i}\}$, $\mathrm{sg}\{Q^{2\pi/3}_{e_i}\} \not\subseteq \mathrm{Sym}_e A'$
7	$A'_{13}, A'_{23}, A'_{11}-A'_{22} = 0$ $A'_{12} = 0$, with $A'_{33} \neq 0$	$O(2) \cong \mathrm{sg}\{Q^{\pi}_{e_1}, Q^{\pi}_{e_2}, Q^{\phi}_{e_3}\} \subseteq \mathrm{Sym}_e A'$ $O(3) \not\subseteq \mathrm{Sym}_e A'$
8	$A'_{33}, A'_{13}, A'_{23} = 0$ $A'_{12}, A'_{11}-A'_{22} = 0$	$O(3) \subseteq \mathrm{Sym}_e A'$

We apply this result, together with (22), (63), Table 2, and Table 3 below.

For orthotropic $\bar{\mathbf{C}}$ with $\mathrm{Sym}_e \bar{\mathbf{C}} = \{Q^{\pi}_{e_1}, Q^{\pi}_{e_2}, Q^{\pi}_{e_3}\}$ the possibilities for $(\mathrm{Sym}_e \bar{A} \cap \mathrm{Sym}_e \bar{B}) \cap \mathrm{Sym}_e \bar{\mathbf{H}}$, and the corresponding restrictions on the parameters describing $\bar{A}, \bar{B}$ and $\bar{\mathbf{H}}$ (in addition to the requirements that $\bar{A}$, $\bar{B}$ be diagonal, and $\bar{\alpha}_1,\ \bar{\beta}_1,\ \bar{\alpha}_3,\ \bar{\beta}_3,\ \bar{\beta}_2,\ \bar{\beta}_4 = 0$), are:

$$
\begin{aligned}
&i) && \{Q^{\pi}_{e_1}, Q^{\pi}_{e_2}, Q^{\pi}_{e_3}\} \cap \{Q^{\pi}_{e_1}, Q^{\pi}_{e_2}, Q^{\pi}_{e_3}\}, \\
& && \quad \bar{A}^{11} \neq \bar{A}^{22},\ \bar{B}^{11} \neq \bar{B}^{22},\ \ \bar{\alpha}_2 \neq 0 \\
&ii) && \{Q^{\pi}_{e_1}, Q^{\pi}_{e_2}, Q^{\phi}_{e_3}\} \cap \{Q^{\pi}_{e_1}, Q^{\pi}_{e_2}, Q^{\pi}_{e_3}\}, \\
& && \quad \bar{A}^{11} = \bar{A}^{22},\ \bar{B}^{11} = \bar{B}^{22},\ \ \bar{\alpha}_2 \neq 0 \\
&iii) && O(3) \cap \{Q^{\pi}_{e_1}, Q^{\pi}_{e_2}, Q^{\pi}_{e_3}\}, \\
& && \quad \bar{A}^{11},\ \bar{A}^{22},\ \bar{B}^{11},\ \bar{B}^{22}\ =\ 0,\ \ \bar{\alpha}_2 \neq 0 \\
&iv) && \{Q^{\pi}_{e_1}, Q^{\pi}_{e_2}, Q^{\pi}_{e_3}\} \cap \{Q^{\pi}_{e_1}, Q^{\pi}_{e_2}, Q^{\pi/2}_{e_3}\}, \\
& && \quad \bar{A}^{11} \neq \bar{A}^{22},\ \bar{B}^{11} \neq \bar{B}^{22},\ \text{and } \bar{\alpha}_2 = 0,\ \bar{\alpha}_4 \neq 0 \\
&v) && \{Q^{\pi}_{e_1}, Q^{\pi}_{e_2}, Q^{\pi}_{e_3}\} \cap \{Q^{\pi/2}_{e_1}, Q^{\pi/2}_{e_2}, Q^{\pi/2}_{e_3}\}, \\
& && \quad \bar{A}^{11} \neq \bar{A}^{22},\ \bar{B}^{11} \neq \bar{B}^{22},\ \text{and } \bar{\alpha}_2 = 0,\ \bar{\alpha}_4 = 5\bar{\alpha}_0 \neq 0 \\
&vi) && \{Q^{\pi}_{e_1}, Q^{\pi}_{e_2}, Q^{\pi}_{e_3}\} \cap \{Q^{\pi}_{e_1}, Q^{\pi}_{e_2}, Q^{\phi}_{e_3}\}, \\
& && \quad \bar{A}^{11} \neq \bar{A}^{22},\ \bar{B}^{11} \neq \bar{B}^{22},\ \text{and } \bar{\alpha}_2,\ \bar{\alpha}_4 = 0 \\
&vii) && \{Q^{\pi}_{e_1}, Q^{\pi}_{e_2}, Q^{\pi}_{e_3}\} \cap O(3) \\
& && \quad \bar{A}^{11} \neq \bar{A}^{22},\ \bar{B}^{11} \neq \bar{B}^{22},\ \text{and } \bar{\alpha}_0,\ \bar{\alpha}_2,\ \bar{\alpha}_4 = 0.
\end{aligned}
\tag{65}
$$

For trigonal $\bar{\mathbf{C}}$ with $\mathrm{Sym}_e\bar{\mathbf{C}} = \{Q^{\pi}_{e_1}, Q^{2\pi/3}_{e_3}\}$ the possibilities for $(\mathrm{Sym}_e\bar{A} \cap \mathrm{Sym}_e\bar{B}) \cap \mathrm{Sym}_e\bar{\mathbf{H}}$, and the corresponding restrictions on the parameters describing $\bar{A}, \bar{B}$ and $\bar{\mathbf{H}}$ (in addition to the requirements that $\bar{A}$, $\bar{B}$ be diagonal, $\bar{A}^{11} = \bar{A}^{22}$, $\bar{B}^{11} = \bar{B}^{22}$, $\bar{\alpha}_1, \bar{\beta}_1, \bar{\alpha}_2, \bar{\beta}_2, \bar{\alpha}_3, \bar{\alpha}_4 = 0, \bar{\beta}_4, \bar{\beta}_3 \neq 0$), are:

$$\begin{aligned} &i) \quad \{Q^{\pi}_{e_1}, Q^{\pi}_{e_2}, Q^{\phi}_{e_3}\} \cap \{Q^{\pi}_{e_1}, Q^{2\pi/3}_{e_3}\}, \\ &ii) \quad O(3) \cap \{Q^{\pi}_{e_1}, Q^{2\pi/3}_{e_3}\}, \quad \bar{A}^{11}, \ \bar{A}^{22}, \ \bar{B}^{11}, \ \bar{B}^{22} = 0. \end{aligned} \tag{66}$$

For tetragonal $\bar{\mathbf{C}}$ with $\mathrm{Sym}_e\bar{\mathbf{C}} = \{Q^{\pi}_{e_1}, Q^{\pi}_{e_2}, Q^{\pi/2}_{e_3}\}$ the possibilities for $(\mathrm{Sym}_e\bar{A} \cap \mathrm{Sym}_e\bar{B}) \cap \mathrm{Sym}_e\bar{\mathbf{H}}$, and the corresponding restrictions on the parameters describing $\bar{A}, \bar{B}$ and $\bar{\mathbf{H}}$ (in addition to the requirements that $\bar{A}$, $\bar{B}$ be diagonal, $\bar{A}^{11} = \bar{A}^{22}$, $\bar{B}^{11} = \bar{B}^{22}$, $\bar{\alpha}_1, \bar{\beta}_1, \bar{\alpha}_2, \bar{\beta}_2, \bar{\alpha}_3, \bar{\beta}_3, \bar{\beta}_4 = 0$, $\bar{\alpha}_4 \neq 0$), are:

$$\begin{aligned} &i) \quad \{Q^{\pi}_{e_1}, Q^{\pi}_{e_2}, Q^{\phi}_{e_3}\} \cap \{Q^{\pi}_{e_1}, Q^{\pi}_{e_2}, Q^{\pi/2}_{e_3}\}, \\ &ii) \quad O(3) \cap \{Q^{\pi}_{e_1}, Q^{\pi}_{e_2}, Q^{\pi/2}_{e_3}\}, \quad \bar{A}^{11}, \ \bar{A}^{22}, \ \bar{B}^{11}, \ \bar{B}^{22} = 0. \end{aligned} \tag{67}$$

For cubic $\bar{\mathbf{C}}$ with $\mathrm{Sym}_e\bar{\mathbf{C}} = \{Q^{\pi/2}_{e_1}, Q^{\pi/2}_{e_2}, Q^{\pi/2}_{e_3}\}$ the only possibility for $(\mathrm{Sym}_e\bar{A} \cap \mathrm{Sym}_e\bar{B}) \cap \mathrm{Sym}_e\bar{\mathbf{H}}$ is $O(3) \cap \{Q^{\pi/2}_{e_1}, Q^{\pi/2}_{e_2}, Q^{\pi/2}_{e_3}\}$, and the corresponding restrictions on the parameters describing $\bar{A}, \bar{B}$ and $\bar{\mathbf{H}}$ are that $\bar{A}, \bar{B} = 0$, $\bar{\alpha}_1, \bar{\beta}_1, \bar{\alpha}_2, \bar{\beta}_2, \bar{\alpha}_3, \bar{\beta}_3, \bar{\beta}_4 = 0$, $\bar{\alpha}_4 = 5\bar{\alpha}_0 \neq 0$.

For transversely isotropic $\bar{\mathbf{C}}$ with $\mathrm{Sym}_e\bar{\mathbf{C}} = \{Q^{\pi}_{e_1}, Q^{\pi}_{e_2}, Q^{\phi}_{e_3}\}$ the possibilities for $(\mathrm{Sym}_e\bar{A} \cap \mathrm{Sym}_e\bar{B}) \cap \mathrm{Sym}_e\bar{\mathbf{H}}$, and the corresponding restrictions on the parameters describing $\bar{A}, \bar{B}$ and $\bar{\mathbf{H}}$ (in addition to the requirements that $\bar{A}$, $\bar{B}$ be diagonal, $\bar{A}^{11} = \bar{A}^{22}$, $\bar{B}^{11} = \bar{B}^{22}$, $\bar{\alpha}_i, \bar{\beta}_i = 0$, $i = 1, \ldots, 4$), are:

$$\begin{aligned} &i) \quad \{Q^{\pi}_{e_1}, Q^{\pi}_{e_2}, Q^{\phi}_{e_3}\} \cap \{Q^{\pi}_{e_1}, Q^{\pi}_{e_2}, Q^{\phi}_{e_3}\}, \\ &\qquad \bar{A}^{11}, \ \bar{B}^{11} \neq 0, \quad \bar{\alpha}_0 \neq 0 \\ &ii) \quad O(3) \cap \{Q^{\pi}_{e_1}, Q^{\pi}_{e_2}, Q^{\phi}_{e_3}\}, \\ &\qquad \bar{A}, \ \bar{B} = 0, \quad \bar{\alpha}_0 \neq 0 \\ &iii) \quad \{Q^{\pi}_{e_1}, Q^{\pi}_{e_2}, Q^{\phi}_{e_3}\} \cap O(3) \\ &\qquad \bar{A}^{11}, \ \bar{B}^{11} \neq 0, \quad \bar{\alpha}_0 = 0. \end{aligned} \tag{68}$$

For isotropic $\bar{\mathbf{C}}$ both $\mathrm{Sym}_e\bar{A} \cap \mathrm{Sym}_e\bar{B}$ and $\mathrm{Sym}_e\bar{\mathbf{H}}$ must be $O(3)$, and so $\bar{A}, \bar{B} = 0$ and $\bar{\alpha}_0, \ \bar{\alpha}_i, \ \bar{\beta}_i = 0, \ i = 1, \ldots, 4$.

5.3. **Computing $\widehat{\bar{\mathsf{C}}}$.** We now compute $\widehat{\bar{\mathsf{C}}} = D * \bar{\mathsf{C}}$ in terms of $\bar{\mathsf{C}}$. We observe that $(D * \bar{\mathsf{C}})^{ijkl} = \bar{\mathsf{C}}^{ijkl}\, a^{e(1)}\, b^{e(2)}\, c^{e(3)}$, where in this formula $e(m) = \delta_{im} + \delta_{jm} + \delta_{km} + \delta_{lm}$ and there is no summation over i, j, k, l. It follows that $D * \bar{\mathsf{H}}$ has components given by

(69)

$ij \backslash kl$	11	22	33
11	$a^4(3\bar\alpha_0 - \bar\alpha_2 + \bar\alpha_4)$	$a^2b^2(\bar\alpha_0 - \bar\alpha_4)$	$-a^2c^2(4\bar\alpha_0 - \bar\alpha_2)$
22		$b^4(3\bar\alpha_0 + \bar\alpha_2 + \bar\alpha_4)$	$-b^2c^2(4\bar\alpha_0 + \bar\alpha_2)$
33			$8c^4\bar\alpha_0$

(70)

$ij \backslash kl$	12	23	13
11	$-a^3b(\bar\beta_2/2 - \bar\beta_4)$	$-a^2bc[\bar\beta_1 - \bar\beta_3]/4$	$-a^3c[3\bar\alpha_1 - \bar\alpha_3]/4$
22	$-ab^3(\bar\beta_2/2 + \bar\beta_4)$	$-b^3c[3\bar\beta_1 + \bar\beta_3]/4$	$-ab^2c[\bar\alpha_1 + \bar\alpha_3]/4$
33	$abc^2\bar\beta_2$	$bc^3\bar\beta_1$	$ac^3\bar\alpha_1$
12	$a^2b^2[\bar\alpha_0 - \bar\alpha_4]$	$-ab^2c[\bar\alpha_1 + \bar\alpha_3]/4$	$-a^2bc[\bar\beta_1 - \bar\beta_3]/4$
23		$-b^2c^2(4\bar\alpha_0 + \bar\alpha_2)$	$abc^2\bar\beta_2$
13			$-a^2c^2(4\bar\alpha_0 - \bar\alpha_2)$

while, writing $M = (\bar\alpha + 2\bar\beta)I + 2(\bar A + 2\bar B)$, $\quad \widehat{\bar{\mathsf{C}}} - (D * \bar{\mathsf{H}})$ is given by:

(71)

$ij \backslash kl$	11	22	33
11	a^4M^{11}	$a^2b^2[\bar\alpha + \bar A^{11} + \bar A^{22}]$	$a^2c^2[\bar\alpha - \bar A^{22}]$
22		b^4M^{22}	$b^2c^2[\bar\alpha - \bar A^{11}]$
33			c^4M^{33}

(72)

$ij \backslash kl$	12	23	13
11	$a^3b(\bar A + 2\bar B)^{12}$	$a^2bc\bar A^{23}$	$a^3c(\bar A + 2\bar B)^{13}$
22	$ab^3(\bar A + 2\bar B)^{12}$	$b^3c(\bar A + 2\bar B)^{23}$	$ab^2c\bar A^{13}$
33	$abc^2\bar A^{12}$	$bc^3(\bar A + 2\bar B)^{23}$	$ac^3(\bar A + 2\bar B)^{13}$
12	$a^2b^2[\bar\beta + \bar B^{11} + \bar B^{22}]$	$ab^2c\bar B^{13}$	$a^2bc\bar B^{23}$
23		$b^2c^2[\bar\beta - \bar B^{11}]$	$abc^2\bar B^{12}$
13			$a^2c^2[\bar\beta - \bar B^{22}]$

To compute the harmonic-Cartan decomposition of $\widehat{\bar{\mathsf{C}}}$ we first compute $\widehat{\bar\alpha} = \alpha_{\widehat{\bar{\mathsf{C}}}}$, $\widehat{\bar\beta} = \beta_{\widehat{\bar{\mathsf{C}}}}$, $\widehat{\bar A} = A_{\widehat{\bar{\mathsf{C}}}}$, $\widehat{\bar B} = B_{\widehat{\bar{\mathsf{C}}}}$, and $\widehat{\bar{\mathsf{H}}} = \mathsf{H}_{\widehat{\bar{\mathsf{C}}}}$, given (51) and (54). We then determine $\widehat{\bar\alpha}_0 = (\alpha_0)_{\widehat{\bar{\mathsf{C}}}}$, $\widehat{\bar\alpha}_i = (\alpha_i)_{\widehat{\bar{\mathsf{C}}}}$, $\widehat{\bar\beta}_i = (\beta_i)_{\widehat{\bar{\mathsf{C}}}}$, $i = 1, \ldots, 4$ from the Cartan decomposition (56) for $\widehat{\bar{\mathsf{H}}}$.

It follows by a calculation, given $\bar A^{33} = -\bar A^{11} - \bar A^{22}$ and $\bar B^{33} = -\bar B^{11} - \bar B^{22}$, that we may write the linear dependence of

$$\widehat{\bar k}_{1313} = (\widehat{\bar A}^{13}, \widehat{\bar B}^{13}, \widehat{\bar\alpha}_1, \widehat{\bar\alpha}_3) \quad \text{on} \quad \bar k_{1313} = (\bar A^{13}, \bar B^{13}, \bar\alpha_1, \bar\alpha_3)$$

$$\text{as} \qquad \widehat{\bar k}_{1313} = K_{1313} \cdot \bar k_{1313},$$

the linear dependence of

$$\widehat{\bar{k}}_{2313} = (\widehat{\bar{A}}^{23}, \widehat{\bar{B}}^{23}, \widehat{\bar{\beta}}_1, \widehat{\bar{\beta}}_3) \quad \text{on} \quad \bar{k}_{2313} = (\bar{A}^{23}, \bar{B}^{23}, \bar{\beta}_1, \bar{\beta}_3)$$
$$\text{as} \qquad \widehat{\bar{k}}_{2313} = K_{2313} \cdot \bar{k}_{2313},$$

and the linear dependence of

$$\widehat{\bar{k}}_{1224} = (\widehat{\bar{A}}^{12}, \widehat{\bar{B}}^{12}, \widehat{\bar{\beta}}_2, \widehat{\bar{\beta}}_4) \quad \text{on} \quad \bar{k}_{1224} = (\bar{A}^{12}, \bar{B}^{12}, \bar{\beta}_2, \bar{\beta}_4)$$
$$\text{as} \qquad \widehat{\bar{k}}_{1224} = K_{1224} \cdot \bar{k}_{1224}.$$

Finally, we write the linear dependence of

$$\widehat{\bar{k}}_{024} = (15\widehat{\bar{\alpha}}, 15\widehat{\bar{\beta}}, 21\widehat{\bar{A}}^{11}, 21\widehat{\bar{A}}^{22}, 21\widehat{\bar{B}}^{11}, 21\widehat{\bar{B}}^{22}, 280\widehat{\bar{\alpha}}_0, 14\widehat{\bar{\alpha}}_2, 8\widehat{\bar{\alpha}}_4)$$
$$\text{on} \quad \bar{k}_{024} = (\bar{\alpha}, \bar{\beta}, \bar{A}^{11}, \bar{A}^{22}, \bar{B}^{11}, \bar{B}^{22}, \bar{\alpha}_0, \bar{\alpha}_2, \bar{\alpha}_4)$$
$$\text{as} \qquad \widehat{\bar{k}}_{024} = K_{024} \cdot \bar{k}_{024}.$$

We observe that the 4×4 matrices K_{1313}, K_{2313}, K_{1224} are invertible. In fact, K_{1313} is given by $ac/28$ times

$$\begin{pmatrix} 4(a^2+5b^2+c^2) & 8(a^2-2b^2+c^2) & -(3a^2+b^2-4c^2) & (a^2-b^2) \\ 4(a^2-2b^2+c^2) & 4(2a^2+3b^2+2c^2) & -(3a^2+b^2-4c^2) & (a^2-b^2) \\ -4(3a^2+b^2-4c^2) & -8(3a^2+b^2-4c^2) & (9a^2+3b^2+16c^2) & -3(a^2-b^2) \\ 28(a^2-b^2) & 56(a^2-b^2) & -21(a^2-b^2) & 7(a^2+3b^2) \end{pmatrix},$$

and has determinant $a^2b^4c^2 \cdot (ac)^4$, up to a positive constant, so does not vanish since $a, b, c > 0$. Similarly, K_{2313} is given by $bc/28$ times

$$\begin{pmatrix} 4(5a^2+b^2+c^2) & 8(-2a^2+b^2+c^2) & -(a^2+3b^2-4c^2) & (a^2-b^2) \\ 4(-2a^2+b^2+c^2) & 4(3a^2+2b^2+2c^2) & -(a^2+3b^2-4c^2) & (a^2-b^2) \\ -4(a^2+3b^2-4c^2) & -8(a^2+3b^2-4c^2) & (3a^2+9b^2+16c^2) & -3(a^2-b^2) \\ 28(a^2-b^2) & 56(a^2-b^2) & -21(a^2-b^2) & 7(3a^2+b^2) \end{pmatrix},$$

and has determinant $a^4b^2c^2 \cdot (bc)^4$, up to a positive constant. In addition, K_{1224} is given by $ab/28$ times

$$\begin{pmatrix} 4(a^2+b^2+5c^2) & 8(a^2+b^2-2c^2) & -2(a^2+b^2-2c^2) & 4(a^2-b^2) \\ 4(a^2+b^2-2c^2) & 4(2a^2+2b^2+3c^2) & -2(a^2+b^2-2c^2) & 4(a^2-b^2) \\ -8(a^2+b^2-2c^2) & -16(a^2+b^2-2c^2) & 4(a^2+b^2+12c^2) & -8(a^2-b^2) \\ 14(a^2-b^2) & 28(a^2-b^2) & -7(a^2-b^2) & 14(a^2+b^2) \end{pmatrix},$$

and has determinant $a^2b^2c^4 \cdot (ab)^4$, up to a positive constant.

For $\bar{\mathbf{C}}$ isotropic, transversely isotropic, cubic, or tetragonal having parameters $(\bar{\alpha},\ \bar{\beta},\ \bar{A}^{11},\ \bar{B}^{11},\ \bar{\alpha}_0,\ \bar{\alpha}_4)$ it follows by the reasoning of Lemma 17 (Section 5.4) that $\widehat{\bar{\mathbf{C}}} = D * \bar{\mathbf{C}}$ may be written in terms of the parameters $\widehat{\bar{\alpha}},\ \widehat{\bar{\beta}},\ \widehat{\bar{A}}^{11}, \widehat{\bar{A}}^{22},\ \widehat{\bar{B}}^{11},\ \widehat{\bar{B}}^{22},\ \widehat{\bar{\alpha}}_0,\ \widehat{\bar{\alpha}}_2,\ \widehat{\bar{\alpha}}_4$.

Finally, we compute K_{024} to be the following. We write $\widehat{\bar{k}}_{024} = (15\widehat{\bar{\alpha}},\ 15\widehat{\bar{\beta}},\ 21\widehat{\bar{A}}^{11},\ 21\widehat{\bar{A}}^{22},\ 21\widehat{\bar{B}}^{11},\ 21\widehat{\bar{B}}^{22},\ 280\widehat{\bar{\alpha}}_0,\ 14\widehat{\bar{\alpha}}_2,\ 8\widehat{\bar{\alpha}}_4)$ as $M \cdot v$, where

$$v \ = \ (\bar{\alpha}, \bar{\beta}, \bar{A}^{11}, \bar{B}^{11}, \bar{\alpha}_0, \bar{\alpha}_4, \bar{\alpha}_0, \bar{\alpha}_0, \bar{\alpha}_0)$$

and M is given by the product of the following two invertible matrices:

$$\begin{pmatrix}
4a^2b^2 & -2a^2b^2 & 4a^2 & -2a^2 & a^4 & 1 & b^4 & -2b^2 & 4b^2 \\
-a^2b^2 & 3a^2b^2 & -a^2 & 3a^2 & a^4 & 1 & b^4 & 3b^2 & -b^2 \\
5a^2b^2 & -4a^2b^2 & 5a^2 & -4a^2 & 2a^4 & -1 & -b^4 & 8b^2 & -10b^2 \\
5a^2b^2 & -4a^2b^2 & -10a^2 & 8a^2 & -a^4 & -1 & 2b^4 & -4b^2 & 5b^2 \\
-2a^2b^2 & 3a^2b^2 & -2a^2 & 3a^2 & 2a^4 & -1 & -b^4 & -6b^2 & 4b^2 \\
-2a^2b^2 & 3a^2b^2 & 4a^2 & -6a^2 & -a^4 & -1 & 2b^4 & 3b^2 & -2b^2 \\
2a^2b^2 & 4a^2b^2 & -8a^2 & -16a^2 & 3a^4 & 8 & 3b^4 & -16b^2 & -8b^2 \\
0 & 0 & 2a^2 & 4a^2 & -a^4 & 0 & b^4 & -4b^2 & -2b^2 \\
-2a^2b^2 & -4a^2b^2 & 0 & 0 & a^4 & 0 & b^4 & 0 & 0
\end{pmatrix}$$

$$\cdot \begin{pmatrix}
1 & 0 & 2 & 0 & 1 & -1 & 0 & 0 & 0 \\
0 & 1 & 0 & 2 & 1 & -1 & 0 & 0 & 0 \\
1 & 0 & -1 & 0 & -4 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 & -4 & 0 & 0 & 0 & 0 \\
1 & 2 & 2 & 4 & 3 & 1 & 0 & 0 & 0 \\
1 & 2 & -4 & -8 & 8 & 0 & 0 & 0 & 0 \\
1 & 2 & 2 & 4 & 0 & 1 & 3 & 0 & 0 \\
0 & 1 & 0 & -1 & 0 & 0 & 0 & -4 & 0 \\
1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -4
\end{pmatrix}.$$

By collecting terms in the above expression we conclude that the (scaled) parameters $(15\widehat{\bar{\alpha}}, 15\widehat{\bar{\beta}}, 21\widehat{\bar{A}}^{11}, 21\widehat{\bar{A}}^{22}, 21\widehat{\bar{B}}^{11}, 21\widehat{\bar{B}}^{22}, 280\widehat{\bar{\alpha}}_0, 14\widehat{\bar{\alpha}}_2, 8\widehat{\bar{\alpha}}_4)$ have the form (24), where the $P(a, b)$ are linearly independent for fixed a, b and are given by

(73)

$$
\begin{array}{rrll}
P_{\bar{\alpha}} = & \Big(6a^4 - & 2[3(a^2-b^2)-4]a^2 & +[1+(a^2-b^2)^2-4(a^2-b^2)], \\
& a^4 - & [(a^2-b^2)+2]a^2 & +[1+(a^2-b^2)^2+(a^2-b^2)], \\
& 6a^4 - & [3(a^2-b^2)+5]a^2 & -[1+(a^2-b^2)^2-10(a^2-b^2)], \\
& 6a^4 - & [9(a^2-b^2)+5]a^2 & -[1-2(a^2-b^2)^2+5(a^2-b^2)], \\
& -a^4 + & 2[2(a^2-b^2)+1]a^2 & -[1+(a^2-b^2)^2+4(a^2-b^2)], \\
& -a^4 - & 2[(a^2-b^2)-1]a^2 & -[1-2(a^2-b^2)^2-2(a^2-b^2)], \\
& 8a^4 - & 8[(a^2-b^2)+2]a^2 & +[8+3(a^2-b^2)^2+8(a^2-b^2)], \\
& - & [2(a^2-b^2)]a^2 & +[(a^2-b^2)^2+2(a^2-b^2)], \\
& & & +[(a^2-b^2)^2]\Big),
\end{array}
$$

$$
\begin{array}{rrll}
P_{\bar{\beta}} = & \Big(2a^4 - & 2[(a^2-b^2)+2]a^2 & +2[1+(a^2-b^2)^2+(a^2-b^2)], \\
& 7a^4 - & [7(a^2-b^2)-6]a^2 & +[2+2(a^2-b^2)^2-3(a^2-b^2)], \\
& -2a^4 + & 4[2(a^2-b^2)+1]a^2 & -2[1+(a^2-b^2)^2+4(a^2-b^2)], \\
& -2a^4 - & 4[(a^2-b^2)-1]a^2 & -2[1-2(a^2-b^2)^2-2(a^2-b^2)], \\
& 5a^4 + & [(a^2-b^2)-3]a^2 & -2[1+(a^2-b^2)^2-3(a^2-b^2)], \\
& 5a^4 - & [11(a^2-b^2)+3]a^2 & -[2-4(a^2-b^2)^2+3(a^2-b^2)], \\
& 16a^4 - & 16[(a^2-b^2)+2]a^2 & +2[8+3(a^2-b^2)^2+8(a^2-b^2)], \\
& - & [4(a^2-b^2)]a^2 & +2[(a^2-b^2)^2+2(a^2-b^2)], \\
& & & [2(a^2-b^2)^2]\Big),
\end{array}
$$

$$
\begin{array}{rrll}
P_{\bar{A}^{11}} = & \Big(12a^4 - & 4[3(a^2-b^2)+2]a^2 & -2[2-(a^2-b^2)^2-2(a^2-b^2)], \\
& 2a^4 - & -2[(a^2-b^2)+2]a^2 & +[4-2(a^2-b^2)^2-(a^2-b^2)], \\
& 12a^4 - & [6(a^2-b^2)-5]a^2 & +2[2-(a^2-b^2)^2-5(a^2-b^2)], \\
& 12a^4 - & [18(a^2-b^2)-5]a^2 & +[4+4(a^2-b^2)^2+5(a^2-b^2)], \\
& -2a^4 + & 2[4(a^2-b^2)-1]a^2 & +2[2-(a^2-b^2)^2+2(a^2-b^2)], \\
& -2a^4 - & 2[2(a^2-b^2)+1]a^2 & +2[2+2(a^2-b^2)^2-(a^2-b^2)], \\
& 16a^4 - & 16[(a^2-b^2)-1]a^2 & +2[16-3(a^2-b^2)^2-4(a^2-b^2)], \\
& - & [4(a^2-b^2)]a^2 & +2[(a^2-b^2)^2-(a^2-b^2)], \\
& & & [2(a^2-b^2)^2]\Big),
\end{array}
$$

(74)
$$
\begin{aligned}
P_{\bar{B}^{11}} = \Big(&4a^4 - 4[(a^2-b^2)-1]a^2 - [8-4(a^2-b^2)^2+2(a^2-b^2)],\\
&14a^4 - 2[7(a^2-b^2)+3]a^2 - [8-4(a^2-b^2)^2-3(a^2-b^2)],\\
&-4a^4 + 4[4(a^2-b^2)-1]a^2 + 4[2-(a^2-b^2)^2+2(a^2-b^2)],\\
&-4a^4 - 4[2(a^2-b^2)+1]a^2 + 4[2+2(a^2-b^2)^2-(a^2-b^2)],\\
&10a^4 + [2(a^2-b^2)+3]a^2 + 2[4-2(a^2-b^2)^2-3(a^2-b^2)],\\
&10a^4 - [22(a^2-b^2)-3]a^2 + [8+8(a^2-b^2)^2+3(a^2-b^2)],\\
&32a^4 + 32[(a^2-b^2)-1]a^2 - 4[16-3(a^2-b^2)^2+4(a^2-b^2)],\\
&- [8(a^2-b^2)a^2] + 4[(a^2-b^2)^2-(a^2-b^2)],\\
&4(a^2-b^2)^2\Big),
\end{aligned}
$$

$$
\begin{aligned}
P_{\bar{\alpha}_0} = \Big(&8a^4 - 8[(a^2-b^2)+2]a^2 + [8+3(a^2-b^2)^2+8(a^2-b^2)],\\
&8a^4 - 8[(a^2-b^2)+2]a^2 + [8+3(a^2-b^2)^2+8(a^2-b^2)],\\
&4a^4 + [5(a^2-b^2)+4]a^2 - [8+3(a^2-b^2)^2+8(a^2-b^2)],\\
&4a^4 - [13(a^2-b^2)-4]a^2 - 2[4-3(a^2-b^2)^2-2(a^2-b^2)],\\
&4a^4 + [5(a^2-b^2)+4]a^2 - [8+3(a^2-b^2)^2+8(a^2-b^2)],\\
&4a^4 - [13(a^2-b^2)-4]a^2 - 2[4-2(a^2-b^2)-3(a^2-b^2)^2],\\
&24a^4 - 24[(a^2-b^2)-8]a^2 + [64+9(a^2-b^2)^2-96(a^2-b^2)],\\
&- [6(a^2-b^2)a^2] - 3[(a^2-b^2)^2+8(a^2-b^2)],\\
&[3(a^2-b^2)^2]\Big),
\end{aligned}
\tag{75}
$$

$$
\begin{aligned}
P_{\bar{\alpha}_4} = \Big(&(a^2-b^2)^2, \quad (a^2-b^2)^2, \quad (a^2-b^2)(2a^2+b^2),\\
&-(a^2-b^2)(a^2+2b^2), \quad (a^2-b^2)(2a^2+b^2),\\
&-(a^2-b^2)(a^2+2b^2), \quad 3(a^2-b^2)^2,\\
&-(a^2-b^2)(a^2+b^2), \quad 8a^4-(a^2-b^2)(7a^2+b^2)\Big).
\end{aligned}
$$

We also compute that $\bar{\mathbf{C}}$ is strong elliptic if $\bar{\alpha}+2\bar{\beta}-2(\bar{A}^{11}+2\bar{B}^{11}+\bar{A}^{22}+2\bar{B}^{22})+8\bar{\alpha}_0>0$. We remark that, due to the fact that D is invertible, if $\bar{\mathbf{C}}$ is strong elliptic, then by the definition of strong ellipticity $\hat{\bar{\mathbf{C}}}$ is also.

5.4. **Characterizing the range of the pullback by D of isotropic, transversely isotropic, cubic, and tetragonal $\bar{\mathbf{C}}$.** We now analyze the possibilities for $\hat{\bar{\mathbf{C}}}$.

Lemma 17. *If* $\mathrm{Sym}_e\bar{\mathbf{C}}$ *contains* $\{Q^\pi_{e_1}, Q^\pi_{e_2}, Q^\pi_{e_3}\}$, *then* $\mathrm{Sym}_e\hat{\bar{\mathbf{C}}}$ *does also.*

Proof. If $\mathrm{Sym}_e\bar{\mathbf{C}}$ contains $\{Q^\pi_{e_1}, Q^\pi_{e_2}, Q^\pi_{e_3}\}$, (i.e., $\bar{\mathbf{C}}$ is orthotropic, tetragonal, cubic, transversely isotropic, or isotropic), then $\hat{\bar{k}}_{1313} = K_{1313}\bar{k}_{1313} = 0$, $\hat{\bar{k}}_{2313} = K_{2313}\bar{k}_{2313} = 0$, $\hat{\bar{k}}_{1224} = K_{1224}\bar{k}_{1224} = 0$; that is, the off-diagonal terms of $\hat{\bar{A}}$ and $\hat{\bar{B}}$ vanish, and $\{Q^\pi_{e_1}, Q^\pi_{e_2}, Q^\pi_{e_3}\} \subseteq \mathrm{Sym}_e\hat{\bar{\mathbf{H}}}$. The result follows from (63) and Table 3. $\square$

Proof of Proposition 14. It is given that there is a polar decomposition of $D\psi^{-1} = SV$ and diagonalization of the stretch factor $S = UDU^T$, $D = \mathrm{diag}\,(a, b, c)$ with $a, b, c > 0$, so that, with respect to the standard basis for Euclidean space, the generators of $\mathrm{Sym}_e\bar{\mathbf{C}}(X') = \mathrm{Sym}_e U^T V * (C \circ \psi)(X')$ are listed in Table 1. It follows that a polar decomposition of $D\psi$ and diagonalization of the stretch factor is given by $V^T UD^{-1}U^T$. Therefore, $(\psi^{-1})^*(\psi^*\mathbf{C}) = (D\psi) * (\psi^*\mathbf{C} \circ \psi^{-1}) = V^T UD^{-1}U^T * (\psi^*\mathbf{C} \circ \psi^{-1})$; $\overline{\psi^*\mathbf{C}}$ in this case is $U^T * (\psi^*\mathbf{C} \circ \psi^{-1})$; and $\mathrm{Sym}_e\overline{\psi^*\mathbf{C}}(X) = \mathrm{Sym}_e U^T * \big(UDU^T V * (\mathbf{C}\circ\psi)\big) \circ \psi^{-1}(X) = \mathrm{Sym}_e D * \bar{\mathbf{C}}(X')$. By Lemma 17 $\mathrm{Sym}_e D * \bar{\mathbf{C}}(X')$ contains $Q^\pi_{e_1}, Q^\pi_{e_2}, Q^\pi_{e_3}$ since $\mathrm{Sym}_e\bar{\mathbf{C}}(X')$ does, where here $\{e_1, e_2, e_3\}$ is the standard basis for Euclidean space. It follows by the presentation of the symmetry groups for anisotropic elastic media given in [C-V-C], pp. 2489-2490, for example, that $\mathrm{Sym}_e D * \bar{\mathbf{C}}(X')$ has generators listed in Table 1. $\square$

It follows for $\bar{\mathbf{C}}$ orthotropic, tetragonal, cubic, transversely isotropic, or isotropic that $\hat{\bar{\beta}}_3 = 0$, and so $\hat{\bar{\mathbf{H}}}$ and $\hat{\bar{\mathbf{C}}}$ are not trigonal.

For $\bar{\mathbf{C}}$ tetragonal, cubic, transversely isotropic, or isotropic we conclude from the representation (24) of $\hat{\bar{k}}_{024}$ that $a = b$ implies $\hat{\bar{A}}^{11} = \hat{\bar{A}}^{22}$, $\hat{\bar{B}}^{11} = \hat{\bar{B}}^{22}$, $\hat{\bar{\alpha}}_2 = 0$, and $\hat{\bar{\alpha}}_4 = a^4\bar{\alpha}_4$. It follows that $\hat{\bar{\mathbf{C}}}$ is not orthotropic. For $\bar{\mathbf{C}}$ isotropic or transversely isotropic $\hat{\bar{\mathbf{C}}}$ is not cubic or tetragonal since $\hat{\bar{\alpha}}_4 = 0$. In fact, for $\bar{\mathbf{C}}$ isotropic $\hat{\bar{\mathbf{C}}}$ is transversely isotropic since $\hat{\bar{\alpha}}_0 = (\bar{\alpha} + 2\bar{\beta})(a^2 - 1)^2/35 \neq 0$ by strong ellipticity. For $\bar{\mathbf{C}}$ cubic (so $\bar{\alpha}_4 = 5\bar{\alpha}_0 \neq 0$) $\hat{\bar{\mathbf{C}}}$ may be tetragonal, depending on the choice of a.

If $\bar{\mathbf{C}}$ is isotropic and $a \neq b$, then $\hat{\bar{A}}^{11} \neq \hat{\bar{A}}^{22}$, $\hat{\bar{B}}^{11} \neq \hat{\bar{B}}^{22}$. Also, $\hat{\bar{\alpha}}_2 = 0$ if and only if $a^2 + b^2$ is $2c^2$. (If $a^2 + b^2 = 2c^2 = 2$, then $\hat{\bar{\alpha}}_4 = (\bar{\alpha} + 2\bar{\beta})(a^2 - 1)^2/2$, which is not zero by strong ellipticity.) It follows that $\hat{\bar{\mathbf{C}}}$ is orthotropic of type *i*) or *iv*).

If $\bar{\mathbf{C}}$ is transversely isotropic, cubic, or tetragonal, and $a \neq b$, then $\hat{\bar{A}}^{11} \neq \hat{\bar{A}}^{22}$, $\hat{\bar{B}}^{11} \neq \hat{\bar{B}}^{22}$, or $\hat{\bar{\alpha}}_2 \neq 0$. If $\hat{\bar{A}}^{11} \neq \hat{\bar{A}}^{22}$ or $\hat{\bar{B}}^{11} \neq \hat{\bar{B}}^{22}$, then $\hat{\bar{\mathbf{C}}}$ is orthotropic of type *i*), or *iv*) – *vii*). On the other hand, if $\hat{\bar{A}}^{11} = \hat{\bar{A}}^{22}$, $\hat{\bar{B}}^{11} = \hat{\bar{B}}^{22}$, $\hat{\bar{\alpha}}_2 \neq 0$, then $\hat{\bar{\mathbf{C}}}$ is orthotropic of type *ii*) or *iii*).

In conclusion, for $\bar{\mathbf{C}}$ isotropic or transversely isotropic, $\widehat{\bar{\mathbf{C}}}$ is orthotropic, transversely isotropic, or isotropic (depending on $\bar{\mathbf{C}}$ and a, b), but will never be tetragonal or cubic; for $\bar{\mathbf{C}}$ tetragonal or cubic, $\widehat{\bar{\mathbf{C}}}$ is (only) orthotropic, tetragonal, or cubic; and for $\bar{\mathbf{C}}$ orthotropic, $\widehat{\bar{\mathbf{C}}}$ is isotropic, transverse isotropic, cubic, tetragonal, or orthotropic, depending on the values of $\bar{\mathbf{C}}$ and a, b.

In fact, in the case that $\bar{\mathbf{C}}$ is isotropic and $a = b$ the range of $D * \bar{\mathbf{C}}$ is a 3-parameter family in the 5-parameter family of transverse isotropic elasticity tensors. In the case that $\bar{\mathbf{C}}$ is isotropic and $a \neq b$ the range of $D * \bar{\mathbf{C}}$ is a 4-parameter family in the 9-parameter family of orthotropic elasticity tensors of type i). In the case that $\bar{\mathbf{C}}$ is cubic and $a = b$ the range of $D * \bar{\mathbf{C}}$ is a 4-parameter family in the 5-parameter family of tetragonal elasticity tensors.

In the case that $\bar{\mathbf{C}}$ is cubic and $a \neq b$ the range of $D * \bar{\mathbf{C}}$ is a 5-parameter family in the 9-parameter family of orthotropic elasticity tensors. In particular, in this case the range of $D * \bar{\mathbf{C}}$ is a 5-parameter family in the 9-parameter family of orthotropic elasticity tensors of type i); is a 4-parameter family in the 7-parameter family of orthotropic elasticity tensors of type ii); is a 4-parameter family in the 8-parameter family of orthotropic elasticity tensors of type iv); is a 3-parameter family in the 7-parameter family of orthotropic elasticity tensors of type v); is a 3-parameter family in the 7-parameter family of orthotropic elasticity tensors of type vi); and is a 2-parameter family in the 6-parameter family of orthotropic elasticity tensors of type vii).

The same reasoning may be applied in the case that $\bar{\mathbf{C}}$ is tetragonal. In general, if $a \neq b$, then the range of $D * \bar{\mathbf{C}}$ is a 6-parameter family in the 9-parameter family of orthotropic elasticity tensors. □

Appendix A. Conditions for Uniqueness for Isotropic Linearly Elastic Media

A.1. **Isotropic media.** Here we list the conditions under which Rachele's uniqueness result for isotropic elastodynamics (Theorem 3) holds.

We denote by $\mathrm{Sec}\,(x, \sigma)$ the sectional curvature of any two-dimensional subspace σ of the cotangent space $T_x M$ of a Riemannian manifold M, for any point x in M. For (x, ξ) in the tangent bundle with $\xi \neq 0$ we set $\mathrm{Sec}\,(x, \xi) = \sup_{\{\xi \in \sigma\}} \mathrm{Sec}\,(x, \sigma)$ and $\mathrm{Sec}^+(x, \xi) = \max\{0, \mathrm{Sec}\,(x, \xi)\}$. By $\partial_- \Omega M$ we denote the collection of all (x, ξ) in the unit tangent bundle of M with $x \in \partial M$ and $\langle \xi, \nu(x) \rangle \leq 0$, where ν is the unit outward normal to $\partial\Omega$. For a Riemannian manifold (M, g) with strictly convex boundary ∂M and with the property that there is no geodesic of infinite length in M, we define

$$k^+(M, g) \;=\; \sup_{(x,\xi) \in \partial_- \Omega M} \int_0^l t \;\; \mathrm{Sec}^+\Big(\gamma(t), \dot{\gamma}(t)\Big)\, dt,$$

where $\gamma : [0, l]$ is the maximal geodesic in $M \cup \partial M$ satisfying the initial (i.e. entry) conditions $\big(\gamma(0), \dot\gamma(0)\big) = (x, \xi) \in \partial_{-}\Omega M$.

For a real covariant tensor field T of rank m on a compact, Riemannian manifold (M, g) the definition of the modulus of T, $|T|_g^2 = g^{i_1 j_1} \ldots g^{i_m j_m} T_{i_1 \ldots i_m} T_{j_1 \ldots j_m}$, is independent of the choice of coordinates. We write $||T||_{C^k(M,g)} = \sum_{l=0}^{k} \sup_{x \in M} |\nabla \ldots \nabla T(x)|_g$ for the C^k-norm of the tensor field T, where ∇ is the covariant derivative and there are l copies of ∇ in the lth term in the definition of $||T||_{C^k(M,g)}$.

We consider an admissible, isotropic, linearly elastic medium $(\Omega, \lambda, \mu, \rho)$ (respectively, $(\Omega, \widehat\lambda, \widehat\mu, \widehat\rho)$) such that $\mu > 0$ and $3\lambda + 2\mu > 0$ (positive definiteness of $\mathbf{C}$) on $\bar\Omega$ and $\lambda = 2\mu$ (i.e. $c_p = 2c_s$) at only isolated points in $\bar\Omega$ (for example). In addition, we assume that the metrics $g_p = ([\lambda + 2\mu]/\rho)^{-1}\,\mathbf{e}$ and $g_s = (\mu/\rho)^{-1}\,\mathbf{e}$ (respectively, $\widehat g_p = ([\widehat\lambda + 2\widehat\mu]/\widehat\rho)^{-1}\,\mathbf{e}$ and $\widehat g_s = (\widehat\mu/\widehat\rho)^{-1}\,\mathbf{e}$) have the property that $g_{p/s}$ does not give rise to caustics; the geodesics of $g_{p/s}$ all exit Ω in finite time; and the geodesics do not graze $\partial\Omega$. Finally, we assume that $\partial\Omega$ is strictly convex with respect to each of g_p, $\widehat g_p$, and $\mathbf{e}$ and that $(\bar\Omega, g_p)$ and $(\bar\Omega, \widehat g_p)$ each have curvature bounded above in the sense that $k^+(\Omega, g_p) < 1/3$ and $k^+(\Omega, \widehat g_p) < 1/3$.

A.2. **Isotropic media with residual stress.** In this section we give the conditions under which Rachele's uniqueness result for isotropic elastodynamics *with residual stress* (Theorem 6) holds.

We consider an admissible, isotropic linearly elastic medium $(\Omega, \lambda, \mu, \rho, \mathcal{T})$ (and another $(\Omega, \widehat\lambda, \widehat\mu, \widehat\rho, \widehat{\mathcal{T}})$) with initial stress $\mathbf{S}$ (respectively, $\widehat{\mathbf{S}}$) given by the symmetric, C^∞ smooth 2-tensor $\mathcal{T}$ ($\widehat{\mathcal{T}}$) which satisfies $\nabla_x \cdot \mathcal{T} = 0$ in Ω (divergence-free in the interior) and $\mathcal{T} \cdot \vec\nu = 0$ on $\partial\Omega$ (zero surface traction), where ν is the unit outer normal to $\partial\Omega$. The same conditions hold for $\widehat{\mathcal{T}}$. (The strong ellipticity condition here for $\mathring{\mathsf{A}}^{aA}{}_b{}^B = \mathring{\mathbf{C}}_{iso}^{A'AB'B} \delta^a{}_{A'} \delta_{bB'} + \delta^a{}_b \mathcal{T}^{AB}$ reduces to the conditions that $\lambda + \mu > 0$ on $\bar\Omega$ and that $\mu(x) I + \mathcal{T}$ be uniformly positive definite on $\bar\Omega$.) In addition, we assume that $\partial\Omega$ is strictly convex with respect to the Euclidean metric; that the normal derivatives of all orders of each of λ, μ, ρ, $\mathcal{T}$, and its counterpart from $\widehat\lambda$, $\widehat\mu$, $\widehat\rho$, $\widehat{\mathcal{T}}$ are the same at $\partial\Omega$; and that the metrics

$$g_p = \Big(\frac{\lambda + 2\mu}{\rho} I + \frac{1}{\rho}\mathcal{T}\Big)^{-1} \quad \text{and} \quad g_s = \Big(\frac{\mu}{\rho} I + \frac{1}{\rho}\mathcal{T}\Big)^{-1}$$

(respectively, $\widehat g_p$ and $\widehat g_s$) satisfy the conditions of Theorem 3. We assume that $(\bar\Omega, g_{p/s})$ each have curvature bounded above in the sense that $k^+(\Omega, g_{p/s}) < 1/4$ and $||R_{g_{p/s}}||_{C^{19}(\Omega, g_{p/s})} \le K_{p/s}$, where R_g is the curvature tensor of the metric g, $K_{p/s} > 0$. Finally, we assume that $\widehat g_{p/s}$ are each close to the Euclidean metric in the sense that for each of them there exists an $\epsilon = \epsilon(K, \Omega, 3) > 0$ so that $||(\widehat g_{p/s})_{ij} - \delta_{ij}||_{C^l(\Omega, e)} < \epsilon$.

Acknowledgments

The authors would like to express their appreciation to the organizers, especially J. McLaughlin and G. Uhlmann, of the program on inverse problems held at the Mathematical Sciences Research Institute, Fall 2001, where joint work on this project was begun. They would also like to acknowledge the continuous support of their Ph.D. advisers, M.E. Taylor and G. Uhlmann.

References

[B] M. Belishev. On a unique recovery of lower-order terms in the wave equation from dynamical boundary measurements, *J. Math. Sci.* **101** (5), 2000, 3408–3421. Dokl. Akad. Nauk SSSR (in Russian), **297** (3), 1987, 524-527.

[C-V-C] P. Chadwick, M. Vianello, and S.C. Cowin. A new proof that the number of linear elastic symmetries is eight. *J. Mech. and Physics of Solids* **49**, 2001, 2471–2492.

[E-M] D.G. Ebin and J.E. Marsden. On the motion of incompressible fluids. Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 2, pp. 211–214. Gauthier-Villars, Paris, 1971.

[F-V] S. Forte and M. Vianello. Symmetry classes for elasticity tensors, *J. Elasticity* **43**, 1996, 81 – 108.

[G] M.E. Gurtin, The linear theory of elasticity, in Handbuch der Physik, Vol. VIa/2, C. Truesdell, ed., Berlin, Springer-Verlag, 1972.

[L-S-U] M. Lassas, V. Sharafutdinov, G. Uhlmann. Semiglobal boundary rigidity for Riemannian metrics, Preprint.

[L-P] Jia Lu and P. Papadopoulos. A covariant constitutive description of anisotropic non-linear elasticity. *Z. Angew. Math. Phys.* **51** (2), 2000, 204–217.

[M-H] Marsden and Hughes, *Mathematical Foundations of Elasticity*, Dover, New York, 1983.

[N-T '96] G. Nakamura and K. Tanuma. A nonuniqueness theorem for an inverse boundary value problem in elasticity. SIAM J. Appl. Math. 56 (2), 1996, 602–610.

[N-T-U] G. Nakamura, K. Tanuma, and G. Uhlmann. Layer stripping for a transversely isotropic elastic medium. SIAM J. Appl. Math. 59 (5), 1999, 1879–1891.

[R I] L. Rachele. Uniqueness of the density in an inverse problem for isotropic elastodynamics, *Trans. AMS*, To appear.

[R II] L. Rachele, An Inverse Problem in Elastodynamics: Determination of the wave speeds in the interior. *J. Diff Eqs.* **162** (2), 2000, 300 – 325.

[R III] L. Rachele, Boundary determination for an inverse problem in elastodynamics. *Comm. PDE.* **25** (11-12), 2000.

[R IV] L. Rachele. Uniqueness in Inverse Problems for Elastic Media with Residual Stress, *Comm. PDE.*, To appear.

[R-K] V.G. Romanov and S.I. Kabanikhin. *Inverse problems for Maxwell's equations.* Inverse and Ill-posed Problems Series. VSP, Utrecht, 1994.

[St-U] P. Stefanov and G. Uhlmann. Rigidity for metrics with the same lengths of geodesics. *Math. Res. Lett.* **5**(1-2), 1998, 83–96.

[Sy-U] J. Sylvester and G. Uhlmann. Inverse problems in anisotropic media. *Contemporary Mathematics* **122** , 1991, 105 - 117.

[Ta I] M.E. Taylor, *Partial Differential Equations, Basic Theory*, Springer, New York, 1996.

[Ta III] M.E. Taylor, *Partial Differential Equations III, Nonlinear Equations*, Springer, New York, 1996.

[U] G. Uhlmann. Inverse boundary value problems and applications, Methodes semi-classiques, Vol. 1 (Nantes, 1991). *Asterisque* **207** (6), 1992, 153 - 211.

[Z-Sp] Q.-S. Zheng and A.J.M. Spencer. Tensors which characterize anisotropies. *Internat. J. Engrg. Sci.* **31** (5), 1993, 679–693.

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