# A NEW PROOF OF THE CHEEGER-GROMOLL SOUL CONJECTURE AND THE TAKEUCHI THEOREM

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Abstract. In this paper, we study the geometry for the evolution of (possibly non-smooth) equi-distant hypersurfaces in real and complex manifolds. First we use the matrix-valued Riccati equation to provide a new proof of the Takeuchi Theorem for peudo-convex Kähler domains with positive curvature. We derive a new monotone principle for *both smooth and non-smooth portions* of equi-distant hypersurfaces in manifolds with nonnegative curvature. Such a new monotone principle leads to a new proof of the Cheeger-Gromoll soul conjecture without using Perelman's flat strip theorem.

In addition, we show that if $M^n$ is a complete, non-compact $C^\infty$-smooth Riemannian manifold with nonnegative sectional curvature, then any distance non-increasing retraction from $M^n$ to its soul $\mathcal{S}$ must be a $C^\infty$-smooth Riemannian submersion, a result obtained independently by B. Wilking.

## Introduction

The classical Oka's Lemma states that if $\Omega$ is a compact pseudoconvex domain with $C^2$-smooth boundary $\partial\Omega$ in the Euclidean space $\mathbb{C}^n$ and if $r(x) = d(x, \partial\Omega)$ is the distance function from boundary, then the complex Hessian of the function $(-\log r)$ is nonnegative on $\Omega$, (i.e., $i\partial\bar\partial[-\log r]|_Q \geq 0$ for all $Q \in \Omega$). For the pseudoconvex domains in a Kähler manifolds with positive curvature, Takeuchi [Ta] (see also Suzuki [Su]) obtained the following result.

**Proposition A.** *(Takeuchi [Ta], [Su]) Let $\Omega$ be a pseudoconvex domain with $C^2$-smooth boundary $\partial\Omega$ in $(M^{2n}, g)$ and let $r = d(x, \partial\Omega)$ be the distance function from $x \in \Omega$ to $\Sigma$. Suppose that the Kähler manifold $(M^{2n}, g)$ has holomorphic bisectional curvature $\geq 2$. Then the equi-distant subdomain $\Omega_{(-t)} = \{x \in \Omega | d(x, \partial\Omega) \geq t\}$ is strictly pseudoconvex for any $t > 0$. Furthermore, we have*

$$i\partial\bar\partial(-\log r)(\zeta, \bar\zeta) \geq \frac{1}{4}\|\zeta\|^2$$

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*for any* $\zeta \in T^{1,0}_x(\Omega)$ *and* $x \in \Omega$.

In this paper we present a new geometric proof of Proposition A using comparison theorems in Riemannian geometry. Another proof of the result above can also be found in Siu [Siu1] via the variational approach. Recently Proposition $A$ has been used by Cao-Shaw-Wang [CaSW] to obtain regularity for the $\bar{\partial}$-Neumann operator on pseudo-convex domains in the complex projective space.

We will use the Riccati equation to provide a new proof of Proposition $A$. The idea of using the matrix-valued Riccati equation has been applied in Riemannian geometry by various authors, including L. Green, Hawking-Ellis, Gromov and Eschenburg, see references in [Es]. Gromov [Gro] emphasized that the evolution of principal curvatures of equi-distance hypersurfaces is the source of the comparison theorems. Our approach is to consider the evolution of *all diagonal entries and diagonal* $2 \times 2$ *sub-matrices* in the matrix-valued Riccati equation (instead of Gromov's method on principal curvatures only).

Let $\{\Sigma_t\}_{a \le t \le b}$ be a family of equi-distant $C^2$-smooth hypersurfaces and let $\sigma : [a, b] \to M^n$ be a geodesic of unit speed orthogonal to each hypersurface $\Sigma_t$ at $\sigma(t)$. The matrix-valued Riccati equation for equi-distant $C^2$-smooth hypersurfaces $\{\Sigma_t\}_{a \le t \le b}$ is given by

$$\mathcal{B}'(t) + [\mathcal{B}(t)]^2 + \mathcal{R}(t) = 0, \tag{0.1}$$

where $\mathcal{R}(t)$ is the curvature matrix and $\mathcal{B}(t)$ is the matrix-representation of the second fundamental form of $\Sigma_t$ with respect to the unit normal vector $\frac{\partial}{\partial t}$ and an orthonormal parallel frame $\{E_i(t)\}_{1 \le i \le n-1}$ along $\sigma$.

We say that two real-valued symmetric matrices $A$ and $B$ satisfy the inequality $A \le B$ if $\langle Av, v\rangle \le \langle Bv, v\rangle$ for any vector $v$.

Under the assumption that the sectional curvatures are nonnegative i.e., $\mathcal{R}(t) \ge 0$, by the Riccati equation (0.1), we derive the following monotone principle for equi-distant hypersurfaces $\{\Sigma_t\}$ in any non-negativity curved manifold $(M^n, g)$.

**Proposition B.** *(Monotone Principle, smooth part) Let* $(M^n, g)$ *be a Riemannian manifold with nonnegative sectional curvature. Suppose that* $\{\Sigma_t\}_{a \le t \le b}$ *is a family of equidistant* $C^2$*-smooth hypersurfaces and* $\{\mathcal{B}(t)\}_{a \le t \le b}$ *is the matrix-representation of the second fundamental form of* $\{\Sigma_t\}$ *with respect to an orthonormal parallel frame* $\{E_j(t)\}_{1 \le i \le n-1}$ *and unit normal direction* $\frac{\partial}{\partial t}$. *Then the following is true.*

*(B.1)* $\mathcal{B}(t)$ *is a monotone matrix-valued function.*

*(B.2) If all sectional curvatures at one point* $p_0 \in \Sigma_{t_0} \subset M^n$ *are strictly positive, then* $\mathcal{B}(t_0)$ *is strictly decreasing at* $p_0$. *More precisely, if* $\mathcal{R}(t) \ge \lambda I > 0$, *then* $\mathcal{B}(t_2) - \mathcal{B}(t_1) \le -\lambda(t_2 - t_1)I < 0$ *for all* $t_2 > t_1$, *where* $I$ *is the identity matrix.*

By a theorem of Siu and Yau, any compact Kähler manifold with positive holomorphic bisectional curvature must be bi-holomorphic to the complex projective space $\mathbb{C}P^n$. By definition, the complex Hessian of a function $f$, $i\partial\bar{\partial} f = i\sum_{j,k=1}^n \frac{\partial^2 f}{\partial z_j \partial \bar{z}_k} dz_j \wedge d\partial\bar{z}_k$, is independent of choice of the metrics on a complex manifold $M^{2n}$. On the other hand, it is well-known (cf. [GW1-2]) that, for any Kähler metric, the complex Hessian is related to the real Hessian as follows:

$$\sqrt{-1}\partial\bar{\partial}f(\tilde{\xi},\bar{\tilde{\xi}}) = \text{Hess}(f)(\xi,\xi) + \text{Hess}(f)(J\xi, J\xi) \tag{0.2}$$

where $J$ is the complex structure of the complex manifold $M^{2n}$, $\xi \in [T(M^{2n})]_{\mathbb{R}}$ and $\tilde{\xi} = \frac{1}{\sqrt{2}}\xi - \sqrt{-1}J\xi$, (see [GW2]). Therefore, it is sufficient to verify Proposition A for domains in $\mathbb{C}P^n$ with respect to the Fubini-Study metric. We will choose $\mathcal{B} = \text{Hess}(-r)|_{\Sigma_{(-r)}}$, where $\Sigma_{(-r)} = \partial\Omega_{(-r)} = \{x \in \Omega | d(x, \partial\Omega) = r\}$. Applying the Riccati equation (0.1) and Proposition B to the matrix-valued function $\mathcal{B} + J^{-1}\mathcal{B}J$, we will get a desired estimate for $i\partial\bar{\partial}(-r)$ in complex tangential directions of $\{\partial\Omega_{(-r)}\}$.

In order to derive estimate of $i\partial\bar{\partial}[-\log r]$ in the other directions, we use a new technique of "upper barrier holomorphic hypersurface" inspired by the work of Calabi [Ca]. In addition, although the function $r(x) = d(x,\partial\Omega)$ may not be differentiable at the cut-loci of $\partial\Omega$ in $\Omega$, we can use the barrier function again to estimate the Hessian of $[-\log r]$ at the cut-locus. Thus, Proposition A will become a consequence of Proposition B. Details are given in Section 1 below.

In the second part of this paper, we are mainly interested in evolutions of equi-distant hypersurfaces (possibly non-smooth) in real Riemannian manifolds. The equi-distant non-smooth convex hypersurfaces played important roles in the Cheeger-Gromoll seminal paper [ChG], which we now briefly describe.

Let $\Omega$ be a compact convex domain in $M$ with possible non-smooth boundary. We consider the equidistant hypersurface $\Sigma_{(-r)} = \{x \in \Omega | d(x, \partial\Omega) = r\}$, which may have singularities as well. When $\Omega$ is convex and has nonnegative sectional curvature, Cheeger and Gromoll [ChG] showed that $\Sigma_{(-r)}$ remains to be a convex real hypersurface, which bounds a convex sub-level set $\Omega_{(-r)} = \{x \in \Omega | d(x, \partial\Omega) \geq r\}$.

Because $\Omega_{(-r)}$ is convex, its outward normal cone is well defined along its boundary $\Sigma_{(-r)}$ as follows:

$$\mathcal{N}^+_Q(\Omega, \partial\Omega) = \{v \in T_Q(M^n) | d(Exp_Q(sv), \Omega) = s \text{ for sufficiently small } s > 0\},$$

where $Exp_p$ is the exponential map of the Riemannian manifold $(M^n, g)$ at $p$.

When $\Omega$ is convex and $(M^n, g)$ has nonnegative sectional curvature, the distance function $r(x) = d(x, \partial\Omega)$ is concave down on the domain $\Omega$, i.e., $\text{Hess}(r) \leq 0$. Sharafutdinov [Sh] further observed that the one-sided gradient of $r$, $\nabla^+ r|_Q$, exists uniquely and never vanishes for $Q \in \Omega$ with $r(Q) < r_{max} = \sup_{P\in\Omega}\{r(P)\}$. Therefore, there is a corresponding one-sided flow given by

$$\frac{d\sigma}{d^+t} = \frac{\nabla^+ r}{\|\nabla^+ r\|^2}(\sigma(t)) \tag{0.3}$$

for $Q \in \Omega$ with $r(Q) < r_{max} = \sup_{P\in\Omega}\{r(P)\}$, where $\frac{d\sigma(t)}{d^+t}$ denotes the right-side derivative of the curve $\sigma$.

Let $\mathbb{P}|_\sigma$ be the parallel transportation along the curve $\sigma$. We say that the normal cones $\{\mathcal{N}^+_{\sigma(t)}(\Omega_{(-t)}, \partial\Omega_{(-t)})\}$ is non-decreasing, if

$$\mathbb{P}|_\sigma[\mathcal{N}^+_{\sigma(t_1)}(\Omega_{(-t_1)}, \partial\Omega_{(-t_1)})] \subset \mathcal{N}^+_{\sigma(t_2)}(\Omega_{(-t_2)}, \partial\Omega_{(-t_2)}),$$

for $t_1 \le t_2$. The following is the non-smooth version of Proposition B.

**Theorem C.** *(Monotone Principle, non-smooth part) Let $(M^n, g)$ be a complete Riemannian manifold with nonnegative sectional curvature. Suppose that $\Omega$ is a convex subdomain with possible non-smooth boundary in $(M^n, g)$. Let $\sigma(t)$ be a trajectory of the one-sided flow (0.3) as above. Then the outward normal cones $\{\mathcal{N}^+_{\sigma(t)}(\Omega_{(-t)}, \partial\Omega_{(-t)})\}$ is non-decreasing along the trajectory $\sigma$.*

*Consequently, the inward tangent cones of $\{T^-(\Omega_{(-t)})\}$ becomes smaller and smaller along any trajectory of the flow (0.3).*

A subset $\Omega$ is said to be *totally convex* in a complete Riemannian manifold $(M^n, g)$, if for any pair of points $\{p, q\} \subset \Omega$ any geodesic $\sigma_{p,q}$ from $p$ to $q$ in $M^n$, we have $\sigma_{p,q} \subset \Omega$.

According to the Cheeger-Gromoll theory [ChG], for a complete nonnegatively curved manifold $(M^n, g)$, there exist a partition $0 = a_0 < a_1 < \dots < a_m < a_{m+1} = \infty$ of $[0, \infty)$ and an exhaustion $M^n = \cup_{t\ge 0}\Omega_t$ such that the following holds:

(0.4.1) $\{\Omega_t\}_{t>a_m}$ is an equi-distant, compact and *totally convex* n-dimensional domains with (possibly non-smooth) boundaries. If $t_2 \ge t_1 \ge a_m$, then $\Omega_{t_1} = \{x \in \Omega_{t_2} | d(x, \partial\Omega_{t_2}) \ge t_2 - t_1\}$;
(0.4.2) For each $m \ge j \ge 1$, the set $\Omega_t = \{x \in \Omega_{a_j} | d(x, \partial\Omega_{a_j}) \ge a_j - t\}$ is totally convex for $t \in [a_{j-1}, a_j]$, where $a_{j-1} = a_j - \max\{d(x, \partial\Omega_{a_j}) | x \in \Omega_{a_j})\}$;
(0.4.3) $\Omega_0 = S$ is the soul of $(M^n, g)$ of dimension $n_0 < n$; In particular, $\Omega_0$ is a totally convex, compact and smooth submanifold without boundary.

Using the flag of the totally convex exhaustion above, Cheeger and Gromoll established the fundamental theory for complete Riemannian manifolds of nonnegative sectional curvature. Among other things, Cheeger and Gromoll [ChG] derived the following important result: *If $(M^n, g)$ is a complete non-compact manifold with nonnegative sectional curvature, then $M^n$ contains a compact totally geodesic submanifold $\mathcal{S}$ without boundary (called a soul of $M^n$) such that $M^n$ is diffeomorphic to the normal vector bundle of $\mathcal{S}$ in $M^n$. In particular, if the soul $\mathcal{S}$ is a point, then $M^n$ is diffeomorphic to the Euclidean space $\mathbb{R}^n$.*

The Cheeger-Gromoll soul conjecture asserts that "if a complete and non-compact Riemannian manifold $(M^n, g)$ has nonnegative sectional curvature and if $M^n$ contains a point $p_0$ where all sectional curvatures are positive, then $M^n$ must be diffeomorphic to the Euclidean space $\mathbb{R}^n$". This is true if $(M^n, g)$ has *positive* sectional curvature *everywhere* by the earlier work of Gromoll and Meyer [GrM]. This conjecture was solved by G. Perelman [Per] by his flat strip theorem. Earlier partial results on the Cheeger-Gromoll soul conjecture were obtained by Marenich, Walschap and Strake, see references in [Per].

Applying Proposition B and Theorem C to Cheeger-Gromoll's totally convex family $\{\Omega_t\}_{0\le t\le\infty}$ described in (0.4.1)-(0.4.3) above, we will prove the following:

**Corollary D.** *(Perelman [Per]) If a complete and non-compact Riemannian manifold $(M^n, g)$ has nonnegative sectional curvature and if $M^n$ contains a point $p_0$ where all sectional curvatures are positive, then the soul $\mathcal{S} = \Omega_0$ of $(M^n, g)$ must be a point. Consequently, $M^n$ is diffeomorphic to the Euclidean space $\mathbb{R}^n$.*

When each leaf of the Cheeger-Gromoll exhaustion $\{\partial\Omega_t\}$ is smooth, the above Corollary D is a direct consequence of Proposition B. To see this, let $\sigma : [0, T] \to M^n$ be a broken geodesic from the soul $\mathcal{S}$ to $p_0$ such that $\sigma'(t)$ is orthogonal to the leaves of the Cheeger-Gromoll exhaustion. Suppose contrary, the soul $\mathcal{S}$ has positive dimension. We consider a piece-wise Jacobi field $\{J(t)\}$ along the broken geodesic $\sigma$ such that $\{J(t)\}$ is continuous with $0 \ne J(0) \in T_{\sigma(0)}(\mathcal{S})$. A direct computation shows that

$$\lambda(t) = \frac{d[\log \|J(t)\|]}{dt} = \mathrm{Hess}(-r)(\frac{J(t)}{\|J(t)\|}, \frac{J(t)}{\|J(t)\|}) = \langle \mathcal{B}\frac{J(t)}{\|J(t)\|}, \frac{J(t)}{\|J(t)\|}\rangle \ge 0$$

due to convexity. Notice that $\lambda(0) = 0$ because the soul $\mathcal{S}$ is totally geodesic. Using Proposition B, one can show that $\lambda(t) \equiv 0$ and $\{J(t)\}$ is a piece-wise *parallel Jacobi field.* When curvature $\mathcal{R} > 0$ is positive, by the Jacobi equation, there is *no* parallel Jacobi field $\{J(t)\}$ along any broken geodesic $\sigma$, where $\sigma$ passes through $p_0$ and $\{J(t)\}$ is orthogonal to $\sigma'(t)$. This is a contradiction and Corollary D follows in this case.

We would like to say a few words about the role of the Riccati equation in our new proof of Corollary D, when the Cheeger-Gromoll exhaustion $\{\partial\Omega_t\}$ is non-smooth. Suppose the contrary is true, the soul $\mathcal{S}$ has positive dimension. In the presence of possible singularities of $\{\partial\Omega_t\}$, we can still construct a piece-wise *parallel Jacobi field* $\{J(t)\}$ as above with the following extra observation.

Even if $Q \in \partial\Omega_t$ is a non-smooth point, the inward tangent cone $T^-_Q(\Omega_t)$ is still well-defined. Similarly, one can define the tangent cone $T^-_Q(\partial\Omega_t)$ of $\partial\Omega$. Notice that the cone $T^-_Q(\partial\Omega_t)$ is not necessarily a linear vector space. A unit vector $\vec{v} \in T^-_Q(\partial\Omega_t)$ is said to be *a regular direction* of $\partial\Omega_t$ at $Q$, if the inverse direction $-\vec{v}$ is tangent to $\partial\Omega_t$ as well, i.e., $\pm\vec{v} \in T^-_Q(\partial\Omega_t)$.

When $\dim(\mathcal{S}) > 0$, we can choose a unit vector $\vec{v} \in T_{Q_0}(\mathcal{S})$. Suppose that $\sigma : [0, \ell] \to M^n$ is a geodesic orthogonal to the soul $\mathcal{S}$ at $\sigma(0) = Q_0$. By the discrete version of Theorem C (Theorem 2.1 below), we conclude that $\pm\mathbb{P}|_\sigma\vec{v}$ remains to be tangent to the leaves of the Cheeger-Gromoll convex exhaustion $\{\partial\Omega_t\}$. I.e., $\pm\mathbb{P}|_\sigma\vec{v}$ remains to be *regular* for each level set $\{\partial\Omega_t\}$. Let $\sigma : [0, T] \to M^n$ be a broken geodesic from the soul $\mathcal{S}$ to $p_0$, which is given by a sequence of the nearest point projections as in [ChG]. Suppose that $\{\vec{V}(t)\}$ is the parallel transportation of $v$ along the broken geodesic $\sigma$. Applying a version of Theorem C (Theorem 2.1 below) several times if needed, we conclude that $\{\pm\vec{V}(t)\}$ is tangent to leaves of $\{\partial\Omega_t\}$. Similarly, we have

$$\lambda(t) = \mathrm{Hess}(-r_j)(\vec{V}(t), \vec{V}(t)) = \langle \mathcal{B}[\vec{V}(t)], \vec{V}(t)\rangle \geq 0$$

due to the convexity of $\{\partial\Omega_t\}$. Clearly, we still have $\lambda(0) = 0$. Using an upper barrier function $\hat{\lambda}$ and a corresponding Riccati equation

$$\hat{\lambda}'(t) + [\hat{\lambda}(t)]^2 + K(t) = 0,$$

we conclude that $0 \leq \lambda(t) \leq \hat{\lambda}(t) \equiv 0$. It follows that $\{\vec{V}(t)\}$ is a piece-wise parallel Jacobi field along $\sigma$, which passes through $p_0$, a contradiction. Hence, Corollary D follows. For details, see section 2 below.

In the Perelman's proof [Per] of Cheeger-Gromoll Soul conjecture, the so-called Sharafutdinov retraction from $M^n$ to its soul $S$ plays an important role. The Sharafutdinov retraction $Sh : M^n \to S$ is produced by the piecewise generalized one-sided gradient flows (0.3) described above, see [Sh] or [Yim]. It is known that the retraction $Sh : M^n \to \mathcal{S}$ is distance non-increasing. However, a-priori, it was not known how smooth the Sharafutdinov retraction would be. L. Guijarro [Gu] has shown that the Sharafutdinov retraction is $C^2$-smooth.

**Theorem E.** *Let $(M^n, g)$ be a complete, non-compact and $C^\infty$-smooth Riemannian manifold with nonnegative sectional curvature. Suppose $\mathcal{S}$ is a soul of $M^n$. Then any distance non-increasing retraction $\Psi : M^n \to \mathcal{S}$ must give rise to a $C^\infty$-smooth Riemannian submersion.*

The $C^\infty$ regularity result in Theorem E is also proved by B. Wilking [Wi] via a different approach independently. If one removes the assumption of *nonnegative curvature*, then the distance non-increasing retraction may not be of $C^2$-smooth, (cf. [BG] or Example 3.14 below).

To prove Theorem E, we show that the distance non-increasing retraction $\Psi$ is compatible with a family of vertical Fermi maps $\{\mathcal{F}_A\}$, where $\mathcal{F}_A$ is related to the exponential map along subset $A$. Precise definitions are given in Section 3.

The organization of this paper goes as follows. We will derive new proofs of the Takeuchi Theorem in Section 1. The proofs of other results in this paper have to deal with possible singularities on each level set. In section 2, we provide a new proof of Cheeger-Gromoll soul conjecture. Section 3 is devoted to the proof of smoothness of the distance non-increasing retractions, in the presence of *nonnegative curvature.* Our new proof of the Takeuchi Theorem inspired the authors to obtain other results of this paper. Our method of using *"upper barrier surfaces"* along with appropriate Riccati equations in this paper seems to be new.

## §1. The evolution of smooth portions of equi-distant hypersurfaces in Kähler manifolds

In this section, we are mainly interested in the evolution of smooth portions of equi-distant hypersurfaces in Kähler domains with nonnegative sectional curvature. As an application of Proposition B (monotone principle for the smooth portions), we provide a new proof for Takeuchi Theorem and Oka Lemma.

Let $M^{2n}$ be a complex manifold with the complex structure J and real dimension 2n. For any $C^2$ smooth function $f$ and a complex vector $\tau$ of (1, 0)-type, the Levi form and complex Hessian are related as follows:

$$\mathcal{L}f(\tau, \bar{\tau}) = 4 \sum_{j,k=1}^{n} \frac{\partial^2 f}{\partial z_j \partial \bar{z}_k} \tau_j \bar{\tau}_k = 2\sqrt{-1}(\partial\bar{\partial} f)(\tau, \bar{\tau}), \tag{1.0}$$

where $\tau = \sum_{j=1}^{n} \tau_j \frac{\partial}{\partial z_j} \in T^{(1,0)}(M^{2n})$. Notice that the complex Hessian $\sqrt{-1}(\partial\bar{\partial} f)$ is independent of the choice of the metrics on $M^n$.

When $M^{2n}$ admits a Kähler metric $g = \langle , \rangle$, both the Levi form $\mathcal{L}f$ and $\sqrt{-1}(\partial\bar{\partial} f)$ are related to the real Hessian of $f$ which we now recall.

Since the Kähler metric $g$ is a Hermitian metric, it preserves the complex structure J, i.e., $|JX|^2 = \langle JX, JX\rangle = \langle X, X\rangle = |X|^2$ for any real vector $X \in [T(M^{2n})]_{\mathbb{R}}$. There is a natural isometry between $T(M^{2n})]_{\mathbb{R}}$ and $T^{(1,0)}(M^{2n})$ over the real numbers. The map

$$u \mapsto \tilde{u} = \frac{1}{\sqrt{2}}(u - \sqrt{-1}Ju) \tag{1.1}$$

is a linear isomorphism from $[T(M^{2n})]_{\mathbb{R}}$ to $T^{(1,0)}(M^{2n})$. Recall that, for $\tilde{u} = \frac{1}{\sqrt{2}}(u - \sqrt{-1}Ju)$, we have

$$\sqrt{-1}\partial\bar{\partial} f(\tilde{u}, \bar{\tilde{u}}) = \mathrm{Hess}(f)(u, u) + \mathrm{Hess}(f)(Ju, Ju)$$

see [GW1]), where $\mathrm{Hess}(f)(X, Y) = XYf - (\nabla_X Y)f = \langle \nabla_X(\nabla f), Y\rangle$ and $\nabla$ is the covariant derivative (the induced connection) determined by the Kähler metric g.

### §1.a. Estimates for the complex Hessian of distance functions

When $f$ has the property $|\nabla f| = |df| = 1$, it is easy to check integral curves of the gradient flow are geodesics of unit speed. Therefore, $\nabla_{\nabla f}(\nabla f) = 0$ and $\mathrm{Hess}(f)(\nabla f, Y) = \langle \nabla_{\nabla f}(\nabla f), Y\rangle = 0$ for any $Y \in [T(M^{2n})]_{\mathbb{R}}$. In particular, if $f(x) = r(x) = d(x, \partial\Omega)$ is a distance function, we have

$$\mathrm{Hess}(r)(\nabla r, Y) = 0, \tag{1.2}$$

for any $Y \in [T(M^{2n})]_{\mathbb{R}}$.

It is sufficient to estimate Hess$(r)$ when it is restricted to the tangential subspace $[T(\partial\Omega_{(-r)})]_{\mathbb{R}}$, where $\Omega_{(-r)} = \{x \in \Omega | d(x, \partial\Omega) \geq r\}$. The real Hessian $\text{Hess}(r)|_{[T(\partial\Omega_{(-r)})]_{\mathbb{R}}}$ is exactly the so-called second fundamental form of $\partial\Omega_{(-r)}$ in the Kähler manifold $(M^{2n}, g)$. It is well-known that the tangential part of Hess(r) satisfies the Riccati equation:

$$\nabla_{\nabla r}\text{Hess}(r) + [\text{Hess}(r)]^2 + \mathcal{R} = 0,$$

where $\mathcal{R}$ is a bi-linear form related to sectional curvatures of the Kähler metric g.

The following result was proved by the variational method (e.g., see Takeuchi [Ta] or Siu [Siu1]). We shall use the Riccati equation to give a new simple proof.

**Theorem 1.1.** *Let $(M^{2n}, g)$ be a Kähler manifold with bisectional curvature $\geq 1$. Let $\Omega \subset M^{2n}$ be a pseudoconvex domain with $C^2$ boundary $\partial\Omega = \Sigma$ and let $\Omega_{(-\epsilon)} = \{x \in \Omega \mid \rho < -\epsilon\}$ for sufficiently small $\epsilon > 0$, where $\rho(x) = -d(x, \Sigma)$. Then*

$$\mathcal{L}(\rho)(\tau, \bar{\tau}) = 2\sqrt{-1}\partial\bar{\partial}(\rho)(\tau, \bar{\tau}) \geq \epsilon\|\tau\|^2, \tag{1.3}$$

*for any $\tau \in T^{(1,0)}(\partial\Omega_{(-\epsilon)})$.*

*Proof.* Let $Q_0 \in \partial\Omega$ and $Exp_{Q_0}$ denote the exponential map from $T_{Q_0}(M^{2n})$ to $M^{2n}$. Let $\sigma : [-t_0, t_0] \to M^{2n}$ be the geodesic given by

$$\sigma(t) = \text{Exp}_{Q_0}(t\nabla\rho), \tag{1.4}$$

for small $t_0 > 0$. We will study how the Levi form $\mathcal{L}(\rho)$ changes along $\sigma$. By (0.2), it suffices to analyze Hess$(\rho)$ along $\sigma(t)$. Recall that $\sigma'(t) = \nabla\rho|_{\sigma(t)}$ and

$$\text{Hess}(\rho)(\nabla\rho, \xi) = \langle \tilde{D}_{\nabla\rho}(\nabla\rho), \xi\rangle \equiv 0$$

since $\sigma$ is geodesic. It remains to discuss Hess$(\rho)(\xi, \xi)$ for $\xi \perp \nabla\rho$, or equivalently, $\xi \in T(\Sigma_{(-s)})$ where $\Sigma_{(-s)} = \{x \in X \mid \rho(x) = -s\} = \partial\Omega_{(-s)}$ for some small number $s > 0$. Notice that the second fundamental form of $\Sigma_{(-s)}$ is equal to the Hess$(\rho)$ restricted to the tangent space $T(\Sigma_{(-s)})$, i.e.,

$$\text{Hess}(\rho)(\xi, \eta) = \langle \nabla_\xi \nabla\rho, \eta\rangle = \text{II}_{\Sigma_{(-s)}}(\xi, \eta)$$

for $\xi, \eta \in T(\Sigma_{(-s)})$.

The second fundamental forms of $\Sigma_{(-s)}$ along $\sigma(s)$ satisfy the Riccati equation. We choose an orthonormal frame $e_1(0), \cdots, e_{2n}(0)$ of $T_{Q_0}(M^{2n})$, where $\tilde{e}_k = \frac{1}{\sqrt{2}}[e_{2k} - \sqrt{-1}Je_{2k}]$, $k = 1, \cdots, n$. We require that $\tilde{e}_1(0), \cdots, \tilde{e}_{n-1}(0)$ span $T^{(1,0)}_{Q_0}(\Sigma)$ and that $e_{2k-1} = -Je_{2k}$. We also choose

$$e_{2n}(0) = \sigma'(0) = \nabla\rho|_{Q_0} \tag{1.5}$$

Let $\{E_k(t)\}$ be a parallel vector field along $\sigma(t)$ with initial condition $E_k(0) = e_k(0)$. Since $X$ is Kähler, we have

$$\begin{aligned} &E_{2n}(t) = \nabla\rho|_{\sigma(t)}, \quad E_{2n-1} = -J(\nabla\rho), \\ &E_{2j-1}(t) = -J(E_{2j}(t)), \quad j = 1, \cdots, n-1 \end{aligned} \tag{1.6}$$

for all $0 \le t \le \epsilon$.

For each $k = 1, \cdots, 2n-1$, we consider the Jacobi field $\xi_k$ with initial condition

$$\begin{cases} \xi_k(0) & = E_k(0), \\ \xi_k'(0) & = \nabla_{E_k(0)}(\nabla\rho). \end{cases}$$

For any Jacobi field $\xi(s)$, we have

$$\begin{aligned} \mathrm{Hess}(\rho)(\xi,\xi) &= \Pi_{\Sigma_{(-s)}}(\xi,\xi) \\ &= \langle \nabla_\xi \nabla\rho, \xi\rangle = \langle \nabla_{\nabla\rho}\xi, \xi\rangle \\ &= \langle \xi'(s), \xi(s)\rangle. \end{aligned} \tag{1.7}$$

Let $A(s) = (a_{jk}(s))$ be the matrix-valued function defined by

$$\xi_k(s) = \sum_{j=1}^{2n-1} a_{jk}(s) E_j(s)$$

and the curvature matrix $R(s) = (R_{ij}(s))$ defined by

$$R(\sigma', E_i)\sigma' = \sum_{j=1}^{2n-1} R_{ji} E_j.$$

With the notation above, we have using the Jacobi equation

$$\begin{aligned} 0 &= \xi_k'' + R(\sigma', \xi_k)\sigma' \\ &= \sum_{j=1}^{2n-1} a_{jk}'' E_j(s) + \sum_{i,j=1}^{2n-1} R_{ji} a_{ik} E_j. \end{aligned}$$

Thus we have the matrix expression of the Jacobi equation

$$A''(s) + R(s)A(s) = 0. \tag{1.8}$$

Let

$$B(s) = A'(s)A^{-1}(s) = (b_{ij}(s)).$$

Then

$$\Pi_{\Sigma_{(-s)}}(E_i, E_j) = b_{ij}(s).$$

Using (1.7), we get

$$\Pi_{\Sigma_{(-s)}}(\xi, \xi) = \langle A'(s)A^{-1}(s)\xi, \xi\rangle = \langle B(s)\xi, \xi\rangle. \tag{1.9}$$

Thus $B(s)$ is the matrix representation of the second fundamental form $\Pi_{\Sigma_{(-s)}}$ with respect to the orthonormal basis $E_1(s), \cdots, E_{2n-1}(s)$. It follows from (1.8) and (1.9) that

$$0 = A''A^{-1} + R = B' + B^2 + R, \tag{1.10}$$

or equivalently,

$$\Pi' + \Pi^2 + R = 0. \tag{1.11}$$

We now apply the above Riccati equation (1.11) to prove Theorem 1.1. If $\tau(s) \in T^{(1,0)}(\Sigma_{(-s)})$, then

$$\tau(s) = \xi(s) - \sqrt{-1}J(\xi(s)),$$

where $\xi = \sum_{k=1}^{2n-2} c_k E_k(s)$ for some $\underset{\sim}{C} = (c_1, \cdots, c_{2n-2}) \in \mathbb{R}^{2n}$.

Let

$$\lambda_\xi(s) = \Pi(\xi(s), \xi(s))$$

and let

$$\mu_\tau(s) = \mathcal{L}(\rho)(\tau(s), \bar{\tau}(s)) = 2i\partial\bar{\partial}(\rho)(\tau(s), \bar{\tau}(s))$$

be the Levi form in the $\tau$ direction. From the assumption that $\Omega$ is pseudoconvex, we have

$$\mu_\tau(0) \geq 0.$$

Using (1.11), we get

$$\begin{aligned} \lambda'_\xi(s) &= \langle B'(s)\underset{\sim}{C}, \underset{\sim}{C}\rangle \\ &= \langle -B^2\underset{\sim}{C}, \underset{\sim}{C}\rangle - \langle R\underset{\sim}{C}, \underset{\sim}{C}\rangle \\ &= -\|B\underset{\sim}{C}\|^2 - \langle R\underset{\sim}{C}, \underset{\sim}{C}\rangle \\ &\leq -\langle R(\sigma', \xi)\sigma', \xi\rangle, \end{aligned} \tag{1.12}$$

where we have used that the second fundamental form is symmetric and $B(s)$ is a symmetric matrix. Similarly, we have

$$\lambda'_{J\xi}(s) \leq -\langle R(\sigma', J\xi)\sigma', J\xi\rangle. \tag{1.13}$$

Substituting (1.12) and (1.13) into (0.2), we obtain

$$\mu'_\tau(s) \leq -(\langle R(\sigma', \xi)\sigma', \xi\rangle + \langle R(\sigma', J\xi)\sigma', J\xi\rangle).$$

The term $(\langle R(\sigma',\xi)\sigma',\xi\rangle + \langle R(\sigma',J\xi)\sigma',J\xi\rangle)$ is equal to the bisectional curvature (see e.g. Zheng [Zh]) of the complex tangent plane spanned by $\tilde{e}_n, \tau$. Thus from our assumption, the bisectional curvature is greater or equal to one. Hence, we have

$$\mu'_\tau(s) \le -1. \tag{1.14}$$

Using

$$\mu_\tau(0) - \mu_\tau(-\epsilon) = \int_{-\epsilon}^{0} \mu'_\tau(s)ds$$

and (1.14), we have that

$$\mu_\tau(-\epsilon) = \mu_\tau(0) - \int_{-\epsilon}^{0} \mu'_\tau(s)ds \ge 0 - (-1)\epsilon = \epsilon$$

for any $0 < \epsilon < t_0$. Thus

$$\mathcal{L}(\rho)|_{\sigma(-\epsilon)}(\tau,\bar{\tau}) \ge \epsilon$$

for any $0 < \epsilon \le t_0$ with $\tau \in T^{(1,0)}(\Sigma_{(-\epsilon)})$. This proves (1.3) and Theorem 1.1. $\square$

We would also like to extend the inequality (1.3) to the subset of full measure in domain $\Omega$, not just near the boundary $\partial\Omega$. To do this, we need to recall the definition of cut loci and focal loci in Riemannian geometry.

**Definition 1.2.** (Cut loci or focal loci) Let $\Omega \subset M^m$ be a compact domain in a Riemannian manifold $(M^m, g)$. Suppose that $\sigma : [0,\ell] \to \Omega$ is a geodesic of unit speed such that $\sigma(0) \in \partial\Omega$ and $\sigma'(0)$ is orthogonal to $\partial\Omega$ at $\sigma(0)$.

(1) The above geodesic segment $\sigma$ is said to be length-minimizing from $\partial\Omega$ if $d(\sigma(t),\partial\Omega) = t$ for any $t \in [0,\ell]$;

(2) Suppose that the above geodesic segment $\sigma$ is length-minimizing from $\partial\Omega$. The endpoint $Q = \sigma(\ell)$ is said to be a cut point of $\partial\Omega$ in $\Omega$ if $d(\sigma(\ell+\epsilon),\partial\Omega) < \ell+\epsilon$ for any $\epsilon > 0$.

(3) The subset of all cut points $Q$ described in (2) is called the cut-loci of $\partial\Omega$ in $\Omega$, denoted by $Cut_\Omega(\partial\Omega)$.

We need to use the following geometric properties of the cut-loci.

**Proposition 1.3.** *([CE] p99, [Pe]) Let $\Omega \subset M^{2n}$ be a compact domain with $C^2$-smooth boundary in a $C^2$-smooth Riemannian manifold $(M^m, g)$. Then*

*(1) The cut-loci of $\partial\Omega$ in $\Omega$ is a closed subset of zero measure;*

*(2) There is a nearest point projection: $\mathcal{P}_{\partial\Omega} : [\bar{\Omega} - Cut_\Omega(\partial\Omega)] \to \partial\Omega$; i.e., for each $Q \notin Cut_\Omega(\partial\Omega)$, there exists the unique nearest point $P_Q = \mathcal{P}_{\partial\Omega}(Q) \in \partial\Omega$ such that $d(Q,\partial\Omega) = d(Q,P_Q)$.*

The proof of Theorem 1.1 also implies the following:

**Corollary 1.4.** *Let $(M^{2n}, g)$ be a Kähler manifold with holomorphic bisectional curvature $\geq 1$. Suppose that $\Omega \subset M^{2n}$ is a pseudoconvex domain with $C^2$ boundary $\partial\Omega = \Sigma$. Let $\Omega_{(-t)} = \{x \in \Omega \mid d(x, \partial\Omega) = d(x, \Sigma) \geq |t|\}$ and $\rho(x) = -r(x) = -d(x, \Sigma)$. Then*

$$\mathcal{L}(\rho)|_Q(\tau, \bar{\tau}) \geq |\rho| \|\tau\|^2 \tag{1.3'}$$

*for any $\tau \in T_Q^{(1,0)}(\partial\Omega_{(\rho)})$ and $Q \notin Cut_\Omega(\partial\Omega)$.*

Notice that neither Theorem 1.1 nor Corollary 1.4 has estimates of complex Hessian $\mathcal{L}(-r)$ on complex normal directions. In fact, we already have $\text{Hess}(r)(\nabla r, Y) = 0$ for any $Y$. Furthermore, one can also construct an example of pseudoconvex domain $\Omega \subset \mathbb{C}$, for which the signed distance function $\rho(x) = \rho_{\partial\Omega}(x)$ has the property $\text{Hess}(\rho)(J\nabla\rho, J\nabla\rho)|_Q < 0$ for some $Q \in \Omega$. In such an example, we have $i\partial\bar{\partial}(\rho)(\widetilde{\nabla\rho}, \overline{\widetilde{\nabla\rho}})|_Q < 0$ for some $Q \in \Omega$.

In order to find a plurisubharmonic function $f$ (i.e., $\mathcal{L}f \geq 0$ on $\Omega$), Oka considers $[-\log r]$ instead of the signed distance function $\rho$. Therefore, in next subsection, we estimate $\mathcal{L}(-\log r)(\tau, \bar{\tau}) = 2i\partial\bar{\partial}[-\log r](\tau, \bar{\tau})$. It will be shown that $\mathcal{L}(-\log r)$ is strictly positive definite in all directions.

## §1.b. The estimates for $i\partial\bar{\partial}(-\log r)$ in all directions

Compact Kähler manifolds with nonnegative holomorphic bi-sectional curvature have been classified, see [Mok]. In particular, Siu and Yau showed that any compact Kähler manifolds with nonnegative holomorphic bi-sectional curvature must be biholomorphic to $\mathbb{C}P^n$. It is sufficient to consider the case of $\mathbb{C}P^n$.

Our goal of this subsection is to show the following

**Theorem 1.5.** *Let $\Omega$ be a pseudoconvex domain with $C^2$ boundary $\partial\Omega = \Sigma$ in $\mathbb{C}P^n$ with the Fubini-Study metric and let $r = d(x, \Sigma)$ be the distance function from $x \in \Omega$ to $\partial\Omega = \Sigma$. Then*

$$\mathcal{L}(-\log r)(\zeta, \bar{\zeta}) = 2i\partial\bar{\partial}(-\log r)(\zeta, \bar{\zeta}) \geq \frac{1}{4}\|\zeta\|^2 \tag{1.15}$$

*for any $\zeta \in T_x^{(1,0)}(\Omega)$ and $x \in \Omega$.*

Before we provide the proof of Theorem 1.5, we need to recall two elementary but useful facts, which will be used in the proof. The first one is related to the definition of Hessian of a continuous function by the barrier functions:

**Fact 1.6** (E. Calabi [Ca]) Let $U$ be an open disk of $\mathbb{C} = \mathbb{R}^2$, $f : U \to \mathbb{R}$ be a real-valued continuous function and $Q_0 \in U$. If there is another $C^2$-smooth function $h : U \to \mathbb{R}$ such that (1) $h \leq f$ on U, (2) $f(Q_0) = h(Q_0)$ and $\triangle h(Q_0) \geq C$, then we have $\triangle f(Q_0) \geq C$.

**Fact 1.7.** Suppose that $\Omega_{(-\epsilon)}$ is strongly pseudo-convex at $P$. Then there exists a small neighborhood $W_\epsilon$ of $P$ and a complex hypersurface $S_{(-\epsilon)} \subset W_\epsilon$ such that

(1.7.1) $S_{(-\epsilon)}$ intersects with $\partial\Omega_{(-\epsilon)}$ at $P$ tangentially, i.e., $[T_P(S_{(-\epsilon)})]_{\mathbb{R}} \subset T_P\big(\partial\Omega_{(-\epsilon)}\big)$;

(1.7.2) $S_{(-\epsilon)}$ lies outside of $\Omega_{(-\epsilon)}$.

For proof of Fact 1.7, see page 46 of [CS]. It was proved in the previous subsection that $\bar{\Omega}_{(-\epsilon)}$ is strongly pseudo-convex for any $\epsilon > 0$, see Theorem 1.1.

*Proof of Theorem 1.5.* We first assume that $x \in U \cap \Omega$, where $U$ is a small neighborhood of $\partial\Omega$. It is easy to see that for any $C^2$ function $f$, we have

$$\mathrm{Hess}(f(\rho))(\xi,\eta) = f'(\rho)\mathrm{Hess}(\rho)(\xi,\eta) + f''(\rho)d\,\rho(\xi) \otimes d\rho(\eta).$$

Let $\rho = -r$. Then

$$\mathrm{Hess}(-\log|\rho|)(\xi,\eta) = \frac{1}{-\rho}\mathrm{Hess}(\rho)(\xi,\eta) + \frac{1}{\rho^2}d\rho(\xi) \otimes d\rho(\eta). \tag{1.16}$$

Using the same notation as in the proof of Theorem 1.1, by (1.3) and (1.16) we already have

$$\mathcal{L}(-\log|\rho|)(\tau,\bar{\tau}) \geq \|\tau\|^2, \quad \tau \in T^{(1,0)}(\partial\Omega_{(-\epsilon)}), \tag{1.17}$$

for $0 < \epsilon < t_0$.

For any $V_{n-1} \in T^{(1,0)}(\partial\Omega_{(-\epsilon)})$ with $|V_{n-1}| = 1$, it remains to estimate (1.15) with $\zeta = aV_{n-1} + b\tilde{e}_n$ for $b \neq 0$.

*Special Case:* $\zeta = \tilde{e}_n$

This part of the proof will be superseded by the proof for the general case below. We include it here, in order to indicate the strategy of our proof for the general case.

From (1.2) and (1.16), we have

$$\mathcal{L}(-\log|\rho|)(\tilde{e}_n,\bar{\tilde{e}}_n) = \frac{1}{|\rho|}\mathrm{Hess}(\rho)(J(\nabla\rho),J(\nabla\rho)) + \frac{1}{\rho^2}. \tag{1.18}$$

Choose $c_0 \in (0, \frac{\pi}{2}]$ such that $\mathcal{B}(0) \geq -2\cot(2c_0)I$. Under the assumption that $\rho = -r$ is $C^2$ and that the sectional curvatures of the metric g are between 1 and 4, by Riccati equation (0.1) it was shown in [Pe] and [Es] that

$$\mathrm{Hess}(-r)(J(\nabla r),J(\nabla(r)) \geq 2\cot[2(r-c_0)], \tag{1.19}$$

for sufficiently small $0 < r < c_0$.

It follows from (1.18) and (1.19) that, for $0 > \rho > -c_0$, we have

$$\begin{aligned}\mathcal{L}(-\log|\rho|)(\tilde{e}_n, \bar{\tilde{e}}_n) &= \frac{1}{|\rho|}\text{Hess}(\rho)(J(\nabla\rho), J(\nabla\rho)) + \frac{1}{\rho^2} \\ &\geq \frac{1}{|\rho|^2}(1 - 2|\rho|\cot(2|\rho + c_0|)) \\ &\geq \frac{1}{|\rho|^2}(1 - 2|\rho|\cot(2|\rho|)).\end{aligned}$$

Set $v = 2|\rho|$ and $h(v) = \frac{1}{v}(1 - v\cot v)$. Since

$$h'(v) = \frac{v^2 - \sin^2 v}{v^2 \sin^2 v} \geq 0$$

and

$$\lim_{v\to 0^+} h(v) = \lim_{v\to 0^+} \frac{1-\cos v}{v\sin v} = \frac{1}{2},$$

we have that

$$\mathcal{L}(-\log|\rho|)(\tilde{e}_n, \bar{\tilde{e}}_n) \geq \frac{2}{|\rho|}h(2|\rho|) \geq \frac{1}{|\rho|}. \tag{1.20}$$

Combining (1.17) and (1.20), we have proved (1.15) for the case either $\zeta = \tilde{e}_n$ or $\zeta = V_{n-1} \in T^{(1,0)}(\partial\Omega_{(-t)})$.

*General Case.*

When $\tau = aV_{n-1} + b\tilde{e}_n$ with $ab \neq 0$ and $V_{n-1} \in T^{(1,0)}(\partial\Omega_{(-t)})$, we observe that

$$\begin{aligned}\text{Hess}(r)(\tau, \bar{\tau}) =& |a|^2\text{Hess}(r)(V_{n-1}, \bar{V}_{n-1}) \\ &+ 2Re\{ab\text{Hess}(r)(V_{n-1}, \bar{\tilde{e}}_n)\} + |b|^2\text{Hess}(r)(\tilde{e_n}, \bar{\tilde{e}}_n).\end{aligned}$$

The term $\text{Hess}(r)(V_{n-1}, \bar{\tilde{e}}_n)$ is very difficult to handle. However, using Facts 1.6-1.7 we will get rid of this term.

Our strategy is as follows: For any given $Q_0 \in \Omega - Cut_\Omega(\Omega)$, we choose a small neighborhood W around $Q_0$ and an upper barrier distance function $\tilde{r} \geq r$. It follows that $-\log r \geq -\log\tilde{r}$ and hence $\text{Hess}(-\log r)|_{Q_0} \geq \text{Hess}(-\log\tilde{r})|_{Q_0}$. When $\tilde{r}(x) = d(x, S)$ for some holomorphic submanifold $S$ of complex dimension $(n-1)$, the Hessian of $\tilde{r}$ has the property that $J\nabla\tilde{r}$ is an eigen-vector of $\text{Hess}(\tilde{r})$. Recall that if $\tilde{r}$ is the distance function, then $\nabla\tilde{r}$ is eigen-vector of $\text{Hess}(\tilde{r})$. In fact, $\text{Hess}(\tilde{r})(\nabla\tilde{r}, .) = 0$. Because $\text{Hess}(\tilde{r})$ is real and symmetric, there is an orthonormal eigen-basis. It follows that

$$\text{Hess}(\tilde{r})(J\nabla\tilde{r}, V_{n-1}) = 0 \tag{1.21}$$

whenever $V_{n-1}$ is orthogonal to $J\nabla\tilde{r}$. The equation (1.21) will play crucial role in the proof presented below.

Let us now carry out the idea above in details.

Motivated by Fact 1.6, we choose $f = -\log r$ and $h(x) = -\log \tilde{r}_S(x)$ where $S$ is a holomorphic submanifold of complex dimension $(n-1)$ and $\tilde{r}_S(x) = d(x, S)$. It remains to construct the complex submanifold $S$ and verify (1.21). For any given $Q_0 \in \Omega$ but $Q_0 \notin Cut_\Omega(\partial\Omega)$, we let $P_0 \in \partial\Omega$ be the nearest point with $d(Q_0, P_0) = d(Q_0, \partial\Omega) = r_0$. Let $\sigma : [0, r_0] \to \bar{\Omega}$ be the geodesic from $P_0$ to $Q_0$.

Let us now apply Fact 1.7. To simplify our proof, we may assume that $\partial\Omega$ is *strongly pseudo-convex* at $P_0$, otherwise, we can use a family of functions $\tilde{r}_\epsilon(x) = d(x, \partial\Omega_{(-\epsilon)})$ instead; and let $\epsilon \to 0$ at the end of our proof.

Using the polar coordinate system around $\sigma(0)$ for the Fubini-Study metric, we see that $\xi_{2n-1}(\tilde{r}) = \sin(2\tilde{r})(J\nabla\tilde{r})$ is a Jacobi field along $\sigma$. Since $\nabla_{\nabla\tilde{r}}\xi = 2\cos(2\tilde{r})(J\nabla\tilde{r})$ is a scalar multiple of $\xi$, we have

$$\nabla_\xi \nabla\tilde{r} = \nabla_{\nabla\tilde{r}}\xi = 2\cot(2\tilde{r})\xi. \tag{1.22}$$

Therefore, the unit direction $J(\nabla\tilde{r}) = \frac{\xi}{|\xi|}$ is an eigenvector of the real symmetric bi-linear form $\mathrm{Hess}(\tilde{r})(X, Y) = \langle \nabla_X \nabla\tilde{r}, Y\rangle$ in $X$ and $Y$. Furthermore, we have

$$\mathrm{Hess}(\tilde{r})(J(\nabla\tilde{r}), Y) = 2\cot(2\tilde{r})\langle J(\nabla\tilde{r}), Y\rangle \tag{1.23}$$

for any tangent vector of $T(\mathbb{C}P^n)$.

Let $\tilde{\Sigma}^{2n-1}_{(-s)} = \{x \in \Omega | d(x, S) = \tilde{r}(x) = s\}$ and we also let $\Re\{\lambda\}$ denote the real part for any complex number $\lambda$. It follows from (1.22) that if $V_{n-1} \in T^{(1,0)}(\tilde{\Sigma}^{2n-1}_{(-s)})$, then

$$\begin{aligned}
&\mathrm{Hess}(\tilde{r})(aV_{n-1} + bJ\nabla\tilde{r}, \overline{aV_{n-1} + bJ\nabla\tilde{r}})\\
&= |a|^2 \mathrm{Hess}(\tilde{r})(V_{n-1}, \bar{V}_{n-1}) + 2\Re\{ab\mathrm{Hess}(\tilde{r})(V_{n-1}, -J\nabla\tilde{r})\}\\
&\quad + |b|^2 \mathrm{Hess}(\tilde{r})(J\nabla\tilde{r}, J\nabla\tilde{r})\\
&= |a|^2 \mathrm{Hess}(\tilde{r})(V_{n-1}, \bar{V}_{n-1}) + 0 + |b|^2 \mathrm{Hess}(\tilde{r})(J\nabla\tilde{r}, J\nabla\tilde{r}),
\end{aligned} \tag{1.24}$$

where the midterm vanishes, because $J\nabla\tilde{r}$ is an eigen-vector of $\mathrm{Hess}(\tilde{r})$ and it is orthogonal to $V_{n-1}$.

It remains to estimate other eigen-values of $\mathrm{Hess}(\tilde{r})$ in complex tangential direction $V_{n-1}$, i.e., we need to estimate $\mathrm{Hess}(\tilde{r})(V_{n-1}, \bar{V}_{n-1})$.

For this purpose, we use the Riccati equation and the same notation as in the proof of Theorem 1.1. Notice that, by the definition of $P_0$, $\sigma$ and our upper barrier function $\tilde{r}$, we see that $\nabla\tilde{r} = \nabla r$ along the geodesic $\sigma$ joining $P_0$ and $Q_0$. We choose an orthonormal frame $\{-Je_2, e_2, \cdots, -Je_{2(n-1)}, e_{2(n-1)}\}$ of $T_{P_0}(S)$, where $S$ is the holomorphic hypersurface of complex dimension $(n-1)$ given by Fact 1.7 above for $\epsilon = 0$.

In what follows, we let $\tilde{\rho} = -\tilde{r}$. Let $\{E_k(t)\}$ be a parallel vector field along $\sigma(t)$ with initial condition $E_k(0) = e_k$. Recall that $e_{2n-1} = -J(\nabla\tilde{\rho})$ and $e_{2n} = \nabla\tilde{\rho}$.

Suppose that $\tilde{B}(s)$ is the matrix representation of the second fundamental form $\Pi_{\tilde{\Sigma}_{(-s)}}$ with respect to the orthonormal basis $E_1(s), \cdots, E_{2n-1}(s)$, where $\tilde{\Sigma}_{(-s)} = \{x \in \Omega | d(x,S) = s\}$. Using the same argument as before, we obtain that

$$0 = \tilde{B}' + \tilde{B}^2 + R.$$

Observe that the proof of Theorem 1.1 is independent of the (2n-1)-th column and the (2n-1)-row of $B(s)$. Since the complex hypersurface is holomorphic, one can show that

$$(i\partial\bar{\partial}\tilde{r})(\tilde{\xi}, \bar{\tilde{\xi}})|_{P_0} = \langle \tilde{B}(0)\xi, \xi\rangle + \langle \tilde{B}(0)J\xi, J\xi\rangle = 0,$$

for $\xi \perp \{\nabla\tilde{r}, J\nabla\tilde{r}\}$. Hence, we have the zero initial condition for $\xi \perp \{\nabla\tilde{r}, J\nabla\tilde{r}\}$ at $P_0$:

$$\langle [\tilde{B}(0) + J^{-1}\tilde{B}(0)J]\xi, \xi\rangle = 0.$$

Replacing the matrix-valued function $B(s)$ by $\tilde{B}(s)$ in the proof of Theorem 1.1, we obtain that if $\tau = \frac{1}{\sqrt{2}}(\xi - iJ\xi) \in T^{(1,0)}(\tilde{\Sigma}_{(-s)})$ then

$$\mathcal{L}(\tilde{\rho})(\tau, \bar{\tau}) = \mathrm{Hess}(\tilde{\rho})(\xi, \xi) + \mathrm{Hess}(\tilde{\rho})(J\xi, J\xi) \geq 2|\rho||\tau|^2, \tag{1.25}$$

where $\tilde{\Sigma}_{(-s)} = \{x \in \Omega | d(x,S) = s\}$. By (1.16), (1.22)-(1.23) and the inequality $\frac{1}{r^2} - \frac{2}{r}\cot(2r) \geq \frac{1}{r}$ above, we obtain

$$[\mathrm{Hess}(-\log\tilde{r})](\tilde{e}_n, \tilde{e}_n) = \frac{1}{r^2} - \frac{2}{r}\cot(2r) \geq \frac{1}{r} \geq \frac{2}{\pi}, \tag{1.26}$$

where we used the fact that $r \leq \mathrm{Diam}(\mathbb{C}P^n)$ and the diameter $\mathrm{Diam}(\mathbb{C}P^n)$ of $\mathbb{C}P^n$ is equal to $\frac{\pi}{2}$.

Let $V_n = \frac{1}{\sqrt{2}}[\nabla r - iJ\nabla r]$ and $V_{n-1} \in T^{(1,0)}_{Q_0}(\partial\Omega_{(-r_0)})$ with $|V_{n-1}| = 1$. Using (1.22)-(1.26) and Fact 1.6, we conclude that, for any $\tau = aV_{n-1} + bV_n \in T^{(1,0)}_{Q_0}(\mathbb{C}P^n)$, the following is true:

$$\begin{aligned}
&\mathcal{L}(-\log r)|_{Q_0}(\tau, \bar{\tau}) \geq \mathcal{L}(-\log\tilde{r})|_{Q_0}(\tau, \bar{\tau}) \\
&= \mathrm{Hess}(-\log\tilde{r})(aV_{n-1} + bV_n, \overline{aV_{n-1} + bV_n}) \\
&= |a|^2\mathrm{Hess}(\log\tilde{r})(V_{n-1}, \bar{V}_{n-1}) + 2\Re\{ab\mathrm{Hess}(-\log\tilde{r})(V_{n-1}, \bar{V}_n)\} \\
&\quad + |b|^2\mathrm{Hess}(-\log\tilde{r})(V_n, \bar{V}_n) \\
&= |a|^2\mathrm{Hess}(\log\tilde{r})(V_{n-1}, \bar{V}_{n-1}) + |b|^2\mathrm{Hess}(-\log\tilde{r})(V_n, \bar{V}_n) \\
&\geq 2|a|^2 + |b|^2[\frac{1}{r^2} - 2\cot(2r)] \geq \frac{1}{2}[|a|^2 + |b|^2] = \frac{1}{2}|\tau|^2
\end{aligned} \tag{1.27}$$

This completes the proof of Theorem 1.5 away from the cut-locus. On the cut-locus $r$ is not $C^2$. However, it is well-known (see Proposition 1.3 above) that the cut-locus of $M^{2n}$ has measure zero in $\Omega$. Observe that, on the cut-locus, the function $r(x) = d(x, \partial\Omega)$ remains to be continuous. By Fact 1.6, one can show that $[-\log r]$ is strictly subharmonic on any complex curve in $\Omega$. Hence, the function $[-\log r]$ is strictly pluri-subharmonic in all of $\Omega$. $\square$

We remark that our proof of Proposition A also gives a new proof of the classical Oka's Lemma. The Oka Lemma states that if $\Omega \subset \mathbb{C}^n$ then the function $(-\log r)$ is pluri-subharmonic in $\Omega$, where $r(x) = d(x, \partial\Omega)$, see [CS] Chapter 3, Theorem 3.4.7 or [Kr] page 117. In this case, the curvature is identically zero. The proof of Theorem 1.1 implies that if $\Omega_{(-\epsilon)} = \{x \in \Omega | d(x, \partial\Omega) \geq \epsilon\}$ then $\partial\Omega_{(-\epsilon)}$ remains to be pseudoconvex. Similarly, the proof of Corollary 1.4 implies that

$$\mathcal{L}(\rho)|_Q(\tau, \bar{\tau}) \geq 0 \tag{1.28}$$

for any $\tau \in T_Q^{(1,0)}(\partial\Omega_{(-t)})$ and $Q \notin Cut_\Omega(\partial\Omega)$.

For the choice of barrier functions $\tilde{r}_{(-\epsilon)}$, it is sufficient to find a complex hypersurface $S_{(-\epsilon)}$ for each $\sigma(-\epsilon)$. This is possible because we have the embedding $\Omega \subset \mathbb{C}^n \subset \mathbb{C}P^n$. By Theorem 1.5, the interior of $\Omega$ has a strictly pseudo-convex exhaustion $\cup\hat{\Omega}_{(-t)}$. We can now use (1.28), Fact 1.6 and Fact 1.7 to complete the proof of Oka's Lemma as in the proof of Theorem 1.5.

## §2. A new monotone principle for non-smooth portions of convex equi-distant hypersurfaces

In this section, we consider the convex equi-distant hypersurfaces in a complete Riemannian manifold with nonnegative sectional curvature.

The proof of Theorem 1.1 inspired us to make the following observation, which is the starting point of our proof of the Cheeger-Gromoll soul conjecture.

### 2.a. Proof for a special case.

The argument in Section 1 also yields the following elementary result, which is a very special case of Cheeger-Gromoll soul conjecture.

**Proposition 2.0.** *Let $(M^n, g)$ be a complete and non-compact Riemannian manifold with nonnegative curvature. Suppose that the Cheeger-Gromoll totally convex exhaustion $M^n = \cup_{j=1}^{m+1} \cup_{a_{j-1} \leq t < a_j} \Omega_t$ described in (0.4.1)-(0.4.3) satisfies the additional property that each relative boundary set $\partial\Omega_t$ is a $C^1$-smooth submanifold for each $t$ (but dimensions of $\{\partial\Omega_t\}$ may jump more than once). Then the Cheeger-Gromoll soul conjecture holds in this case.*

*Proof.* By the assumption in the Cheeger-Gromoll soul conjecture, $(M^n, g)$ is a complete and non-compact Riemannian manifold with nonnegative sectional curvature and $M^n$ contains a point $p_0$ where all sectional curvatures are positive. Let $W = \{x \in M^n |$ all sectional curvatures are positive at $x\}$ be a subset of $M^n$. Clearly, $W$ is relatively open in $M^n$. Hence, $\dim(W) = \dim(M^n) = n$. Let $\{\Omega_t\}$ be the Cheeger-Gromoll totally convex exhaustion described in (0.4.1)-(0.4.3). Notice that if $t \le a_m$ then $\dim(\Omega_t) \le \dim(\Omega_{a_m}) \le n-1$. Thus, we have $[W \setminus \Omega_{a_m}] \neq \emptyset$ is a non-trivial set. We may assume that $p_0 \in \partial\Omega_{t_{m+1}}$ for some $t_{m+1} > a_m$.

By our assumption that each relative boundary set $\partial\Omega_t$ is a $C^1$-smooth submanifold. Hence, the function $r_{m+1}(x) = d(x, \Omega_{a_m})$ is $C^1$-smooth at $x \notin \Omega_{a_m}$. Similarly, the function $r_j(x) = d(x, \partial\Omega_{a_j})$ is $C^1$-smooth at $x \in \Omega_{a_j} - \Omega_{a_{j-1}}$. The semi-flow (0.3) becomes piecewise $C^1$-smooth. Because $\|\nabla r_j\|(x) = 1$, then each trajectory of the dynamic system $\frac{dx}{dt} = \nabla r_j(x(t))$ is a local length-minimizing geodesic segment between level sets of $r_j$. Hence, each trajectory of the flow (0.3) is a geodesic segment.

More precisely, let $\mathcal{P}_m : M^n \to \Omega_{a_m}$ be the nearest projection along the trajectory of flow (0.3). Similarly, let $\mathcal{P}_j : \Omega_{a_{j+1}} \to \Omega_{a_j}$ be the nearest point projection as well, for $j = 0, 1, ..., m-1$. We choose $Q_{m+1} = p_0$, $Q_m = \mathcal{P}_m(Q_{m+1})$ and $Q_j = \mathcal{P}_j(Q_{j+1})$ for $j = m-1, m-2, ..., 0$. Let $\sigma_j : [0, \ell_j] \to M^n$ be the geodesic segment of unit speed from $Q_{j-1}$ to $Q_j$ for $j = 1, ...., m$. Clearly, $\cup_{j=1}^{m+1}\sigma_j$ is a continuous broken geodesic.

To simplify our proof, we assume that $Q_j \neq Q_{j-1}$ for all $j$. Choose $t_j = \inf\{t | Q_j \in \Omega_t\}$. We now use Proposition B (or Theorem 1.10 of [ChG, p420-421]) to show that the soul $\mathcal{S}$ must be a point.

Suppose contrary, $\dim(\mathcal{S}) = k > 0$. Choose a unit vector $w \in T_{Q_0}(\mathcal{S})$ and let $c_0(s) = Exp_{Q_0}(sw)$. Let $V_1(0) = \sigma'(0)$ and $\{V_1(s)\}_{s\in\mathbb{R}}$ be the parallel transport of $V_j(0)$ along the geodesic $c_0$.

Proposition B (or Theorem 1.10 of [ChG, p420-421]) implies that the strip $\phi_1 : \mathbb{R} \times [0, a_1] \to \Omega_{a_1}$ defined by $\phi_1(s,t) = Exp_{c_0(s)}[tV_j(s)]$ is flat and totally geodesic. Consequently, the curve $c_1 : \mathbb{R} \to \partial\Omega_{t_1}$ defined by $c_1(s) = \phi_1(s, t_1)$ is geodesic.

Using Proposition B again and induction on $j$, for each $j$ we get a totally geodesic flat strip

$$\begin{aligned} \phi_j : \quad & \mathbb{R} \times [a_{j-1}, t_j] \to M^n \\ & (s,t) \to Exp_{c_j(s)}[(t - a_{j-1})V_j(s)] \end{aligned} \tag{2.0.j}$$

for $j = 1, ...m$, where $c_j(s) = \phi_{j-1}(s, t_{j-1})$ and $\{V_j(s)\}$ is the parallel vector field along $c_j$ with $V_j(0) = \sigma_j'(0)$. Consequently, because the strip is totally geodesic and flat, we would have

$$K(\frac{\partial\phi_{m+1}}{\partial s}, \frac{\partial\phi_{m+1}}{\partial t}) = \langle R(\frac{\partial\phi_{m+1}}{\partial t}, \frac{\partial\phi_{m+1}}{\partial s})\frac{\partial\phi_{m+1}}{\partial t}, \frac{\partial\phi_{m+1}}{\partial s}\rangle = 0,$$

where $R(X,Y)Z = -\nabla_X\nabla_Y Z + \nabla_Y\nabla_X Z + \nabla_{[X,y]}Z$ is the curvature tensor. This is contradict to the assumption that $M^n$ has positive curvature in all directions at $p_0$. Thus, $\dim(\mathcal{S}) = 0$.

The case of $Q_{j_0} = Q_{j_0+1}$ for some $j_0$ can be handled similarly with minor modifications. This completes the proof of Proposition 2.0. $\square$

**2.b. Proof of the soul conjecture assuming Theorem 2.1.**

Let us now turn our attention to the general case of Cheeger-Gromoll convex exhaustion $\{\partial\Omega_t\}_{t\geq 0}$ with possible singularities.

Inspired by Proposition 2.0 and its proof, we make the following observation. In order to prove Cheeger-Gromoll soul conjecture, we *only* need to construct a sequence of totally geodesic strips along the trajectory of the semi-flow (0.3) if the soul $\mathcal{S}$ has positive dimension.

For this purpose, we need to point out the main difficulties in the construction of flap strips. In the proof of Proposition 2.0, we used the extra assumption that the function $r_j(x) = d(x, \partial\Omega_{a_j})$ satisfies $\|\nabla r_j\| = 1$ on $\Omega_{a_j}$. However, the equality $\|\nabla r_j\| = 1$ fails when $\partial\Omega_{a_j}$ has singularities. For example, if $\partial\Omega_{a_j}$ is a triangle in Euclidean plane $\mathbb{R}^2$ (cf. Example 2.4 below). In fact, if $\Omega$ has non-smooth boundary then it is not necessarily true that $U_\epsilon(\Omega_{(-\epsilon)}) = \Omega$, where $\Omega_{(-\epsilon)} = \{x \in \Omega | d(x, \partial\Omega) \geq \epsilon\}$ and $U_\epsilon(A)$ is the $\epsilon$-neighborhood of $A$.

When the Cheeger-Gromoll exhaustion $\{\partial\Omega_t\}$ has possible singularities, we will use a version of Theorem C and the argument in previous sub-section to prove the Cheeger-Gromoll soul conjecture as follows.

Notice that in the proof of Proposition 2.0, we used the nearest point projection. Hence, we need to modify Theorem C in terms of the nearest point projection. In addition, Sharafutdinov and Yim observed that the trajectories of the semi-flow (0.3) can be approximated by a sequence of the nearest point projections. In what follows, we consider the nearest point projection instead. Here is a version of Theorem C, which we will need to construct totally geodesic flat strips.

**Theorem 2.1.** *(Discrete version of Theorem C) Let $\{\Omega_t\}$ be the Cheeger-Gromoll totally convex exhaustion for an open complete manifold $(M^n, g)$ with nonnegative sectional curvature be as in (0.4.1)-(0.4.3). Suppose that, for any sufficiently large $T > 0$, there exists a $\delta_T > 0$ such that whenever $a_{j-1} \leq t_0 < t \leq a_j \leq T$ with $t - t_0 < \delta_T$, the nearest point projection $\mathcal{P} : \Omega_t \to \Omega_{t_0}$ is well-defined. Let $\sigma : [0, \ell] \to M^n$ be a length-minimizing geodesic segment from $Q_{t_0} = \mathcal{P}(Q_t)$ to $Q_t \in \partial\Omega_t$, where $\sigma(u) = Exp_{Q_{t_0}}[u\sigma'(0)]$. Then the outward normal cones satisfy*

$$\mathbb{P}_{-\sigma}[\mathcal{N}^+_{Q_t}(\Omega_t, \partial\Omega_t)] \subset \mathcal{N}^+_{Q_{t_0}}(\Omega_{t_0}, \partial\Omega_{t_0}), \tag{2.1}$$

*where $\mathbb{P}_{-\sigma}$ is parallel transport along $-\sigma$ and $-\sigma(t) = \sigma(-t)$*

*In other words, the inward tangential cones becomes smaller and smaller as $t \searrow t_0^+$, i.e.,*

$$\mathbb{P}_{-\sigma}[T^-(\bar{\Omega}_t)\}] \supset [T^-(\bar{\Omega}_{t_0})\}], \tag{2.1'}$$

*for $t \geq t_0$.*

Using Theorem 2.1 and proof of Proof of Proposition 2.0, we obtain the following broken flat strip theorem.

**Proposition 2.2.** *Suppose that Theorem 2.1 holds and suppose that the soul $\mathcal{S}$ has dimension $\geq 1$. Then, for each $p \in M^n$, there is a broken totally geodesic flat strips $\{\phi_0, ...., \phi_k\}$ with $\phi_j : \mathbb{R} \times [0, u_j] \to M^n$ such that*

*(2.2.1) $c_0(\mathbb{R}) = \phi_0(\mathbb{R}, 0)$ is a geodesic in the soul $\mathcal{S}$;*

*(2.2.2) Two consecutive flat strips $\phi_{j-1}$ and $\phi_j$ meet at a common geodesic $c_j$, for $j = 1, ..., k$; I.e, $\phi_{j-1}(s, u_{j-1}) = \phi_j(s, 0) = c_j(s)$;*

*(2.2.3) $c_k(\mathbb{R}) = \phi_k(\mathbb{R}, u_k)$ is a geodesic passing through $p$.*

*Consequently, the Cheeger-Gromoll soul conjecture is true, assuming Theorem 2.1.*

*Proof.* Choose $T$ such that $p \in \partial\Omega_T$. If $\{\cup_{i=1}^{m+1} \cup_{a_{i-1} \leq t < a_i} \Omega_t\}$ is the flag of Cheeger-Gromoll convex exhaustion. Let $i_0 = \max\{i | a_i \leq T\}$. We choose a refined partition $0 = t_0 < t_1 < .... < t_k = T$ such that $\{t_1, ..., t_k\} \supset \{a_0, a_1, ..., a_{i_0}, T\}$ and $|t_{j-1} - t_j| < \delta_T$ for $j = 1, ...k$, where $\delta_T$ is as in Theorem 2.1.

Suppose that $\mathcal{P}_j : \Omega_{t_j} \to \Omega_{t_{j-1}}$ is the nearest point projection. Define $Q_k = p$, $Q_{k-1} = \mathcal{P}_k(Q_k)$ and $Q_{j-1} = \mathcal{P}_j(Q_j)$ inductively on $j$. Consider the geodesic segment $\sigma_j : [0, u_j] \to M^n$ of unit speed from $Q_{j-1}$ to $Q_j$.

When the soul $\mathcal{S}$ has positive dimension, we choose a geodesic $c_0 : \mathbb{R} \to \mathcal{S}$ with $c_0(0) = Q_0$ and $|c_0'(0)| = 1$. Let $\{V_1(s)\}$ be a parallel vector field along $c_0$ such that $V_1(0) = \sigma_1'(0)$. We now consider the following Fermi map.

$$\begin{aligned} \phi_1 : \quad & \mathbb{R} \times [0, u_1] \to M^n \\ & (s, u) \to Exp_{c_0(s)}[uV_1(s)] \end{aligned} \tag{2.2.1}$$

Using the convexity of $\{\Omega_t\}$ and the assumption of nonnegative sectional curvature, we will verify the following:

**Claim 2.3.** Let $(M^n, g)$, $\{\Omega_t\}$, $\{\mathcal{P}_j\}$, $Q_j$, $\sigma_j$ and $\phi_1$ be as above. Then

(2.3.1) The map $\phi_1$ is a totally geodesic isometric immersion;

(2.3.2) The vector field $\frac{\partial \phi_1}{\partial s}(u, s)$ is a parallel Jacobian field along the geodesic $u \to \phi_1(s, u)$;

(2.3.3) The sectional curvature of the strip $\phi_1(\mathbb{R} \times [0, u_1])$ is zero:

$$K(\frac{\partial \phi_1}{\partial s}, \frac{\partial \phi_1}{\partial u}) = \langle R(\frac{\partial \phi_1}{\partial u}, \frac{\partial \phi_1}{\partial s})\frac{\partial \phi_1}{\partial u}, \frac{\partial \phi_1}{\partial s}\rangle = 0.$$

To verify Claim 2.3, for each $\hat{s} \in \mathbb{R}$, we let $\{W_{\hat{s}}(u)\}$ be a parallel vector field along the vertical geodesic

$$\eta_{\hat{s}} : u \to \phi_1(\hat{s}, u) \tag{2.3}$$

with $W_{\hat{s}}(u) = \frac{\partial \phi_1}{\partial s}(\hat{s}, 0) = c_0'(\hat{s})$.

It follows from Theorem 2.1 that if $[\pm W_{\hat{s}}(0)] \in T^-_{\eta_{\hat{s}}(0)}(\bar{\Omega}_{t(0)}) = T^-_{\eta_{\hat{s}}(0)}(\mathcal{S})$ then $[\pm W_{\hat{s}}(u)] \in T^-_{\eta_{\hat{s}}(u)}(\bar{\Omega}_{t(u)})$ as well, for all $u \in [0, u_1]$. Recall that by Cheeger-Gromoll's convex exhaustion, the inward tangent cone $T^-_{\eta_{\hat{s}}}(\bar{\Omega}_{t(u)})$ is convex. By the convexity and the fact that $[\pm W_{\hat{s}}(u)] \in T^-_{\eta_{\hat{s}}(u)}(\bar{\Omega}_{t(u)})$, we conclude that $[\pm W_{\hat{s}}(u)] \in T^-_{\eta_{\hat{s}}}(\partial \Omega_{t(u)})$.

Since $[\pm W_{\hat{s}}(u)] \in T^-_{\eta_{\hat{s}}}(\partial \Omega_{t(u)})$, we can consider the horizontal Fermi map

$$\begin{aligned} \Psi_{1,\hat{s}} : \quad & [0, u_1] \times \mathbb{R} \to M^n \\ & (s, u) \to Exp_{\eta_{\hat{s}}(u)}[sW_{\hat{s}}(u)], \end{aligned}$$

which is compatible with the 1-parameter family of curves $\{\gamma_u\}$ which we now describe.

Let $\hat{\Sigma}_{1,\hat{s}} = \Psi_{1,\hat{s}}\big([0, u_1] \times (-\epsilon, \epsilon)\big)$ be an immersed surface. Suppose that the curve $\gamma_u = \hat{\Sigma}_{1,\hat{s}} \cap \partial \Omega_{t(u)}$ is the intersection curve passing thorough $\eta_{\hat{s}}(u)$. Let us choose a parametrization $\gamma_u : (-\epsilon, \epsilon) \to \hat{\Sigma}_{1,\hat{s}}$ such that $\gamma_u(0) = \eta_{\hat{s}}(u)$. The argument above implies that the $\gamma_u$ has the tangential vector $W(u)$ at $\gamma_u(0)$.

Because of the convexity of $\partial \Omega_{t(u)}$ with respect to the outward normal vector $\eta'_{\hat{s}}(u)$, the curve $\gamma_u$ is convex at $\gamma_u(0)$ as well. Thus, $\gamma_u$ has nonnegative geodesic curvature $\lambda(u) \geq 0$ at the point $\gamma_u(0) = \eta_{\hat{s}}(u)$.

By our construction, *along the curve $\eta_{\hat{s}}$, the $\hat{\Sigma}_{1,\hat{s}}$ is totally geodesic.* (To see this, one computes the vector-valued the second fundamental form $\mathrm{II}_{\hat{\Sigma}_{1,\hat{s}}}$ of the surface $\hat{\Sigma}_{1,\hat{s}}$ along the curve $\eta_{\hat{s}}$ as follows. It is easy to see that $\nabla_{\frac{\partial \Psi_{1,\hat{s}}}{\partial u}} \frac{\partial \Psi_{1,\hat{s}}}{\partial u}(u, 0) = \nabla_{\frac{\partial \Psi_{1,\hat{s}}}{\partial u}} \frac{\partial \Psi_{1,\hat{s}}}{\partial s}(u, 0) = \nabla_{\frac{\partial \Psi_{1,\hat{s}}}{\partial s}} \frac{\partial \Psi_{1,\hat{s}}}{\partial s}(u, 0) = 0$. Hence we have $\mathrm{II}_{\hat{\Sigma}_{1,\hat{s}}}(X, Y)|_{(u,0)} = 0$ for all $u \in [0, u_1]$). It follows that, along the curve $\eta_{\hat{s}}$, the intrinsic curvature $K_{\hat{\Sigma}_{1,\hat{s}}}$ of $\hat{\Sigma}_{1,\hat{s}}$ is equal to its extrinsic curvature:

$$K(u) = K_{\hat{\Sigma}_{1,\hat{s}}}(u, 0) = K_{\hat{\Sigma}_{1,\hat{s}}}(\eta'_{\hat{s}}, W_{\hat{s}}(u)) = K_{M^n}(\eta'_{\hat{s}}(u), W_{\hat{s}}(u)) \geq 0. \tag{2.4}$$

Notice that the geodesic segment $\eta_{\hat{s}}$ is orthogonal to the 1-parameter family of curves $\{\gamma_u\}_{0 \leq u \leq u_1}$. However, $\{\gamma_u\}_{0 \leq u \leq u_1}$ is not necessarily an equi-distance family of curves in the surface $\hat{\Sigma}_{1,\hat{s}}$. To overcome this difficulty, we will replace $\{\gamma_u\}$ by a family of "upper barrier" curves $\{\hat{\gamma}_u\}$ as follows. Let us compare two distance functions:

$$r(x) = d_{M^n}(x, c_0(\mathbb{R})) \text{ and } \hat{r}(x) = d_{\hat{\Sigma}_{1,\hat{s}}}(x, c_0(\mathbb{R}))$$

for $x \in \hat{\Sigma}_{1,\hat{s}}$.

Clearly, $\hat{r}(x) \geq r(x)$ for $x \in \hat{\Sigma}_{1,\hat{s}}$. Moreover, $\hat{r}(x) = r(x)$ when $x$ is on the geodesic $\eta_{\hat{s}}$. Let $\hat{\lambda}(u) = \mathrm{Hees}_{\hat{\Sigma}}(\hat{r})(W_{\hat{s}}(u), W_{\hat{s}}(u))|_{\eta_{\hat{s}}(u)}$. By Fact 1.6, we have

$$\hat{\lambda}(u) \geq \lambda(u) \geq 0. \tag{2.5}$$

By a version of (0.1), we have the Riccati equation for $\hat{\lambda}(u) = \mathcal{B}(u)$ along the orthogonal geodesic $\eta_{\hat{s}}$ as follows:

$$\frac{\partial \hat{\lambda}(u)}{\partial u} + [\hat{\lambda}(u)]^2 + K = 0. \tag{0.1'}$$

Recall that by assumption and (2.5) we have

$$\hat{\lambda}(0) = 0 \quad \text{ and } \hat{\lambda}(u) \geq 0. \tag{2.6}$$

It follows from (2.4), (0.1') and (2.6) that $\hat{\lambda}(u) \equiv 0$ for all $u \in [0, u_1]$. Using (0.1') and the fact that $\hat{\lambda}(u) = 0$ for all u, we conclude

$$K_{M^n}(\eta'_{\hat{s}}(u), W_{\hat{s}}(u)) = K_{\hat{\Sigma}_{1,\hat{s}}}(\eta'_{\hat{s}}(u), W_{\hat{s}}(u)) = 0, \tag{2.7}$$

for all $u \in [0, u_1]$.

It remains to show that $W_{\hat{s}}(u) = \frac{\partial \phi_1}{\partial s}(\hat{s}, u)$. For this purpose, we observe that the bi-linear symmetric curvature form $R_{(\hat{s},u)} : (X, Y) \to \langle R_{M^n}(\eta'_{\hat{s}}(u), X)\eta'_{\hat{s}}(u), Y\rangle$ is nonnegative semi-definite in $\{X, Y\}$. By (2.7), we know that $\{W_{\hat{s}}(u)\}$ is an eigenvector of the symmetric bi-linear form $R_{(\hat{s},u)}$ with eigenvalue 0:

$$R_{M^n}(\eta'_{\hat{s}}(u), W_{\hat{s}}(u))\eta'_{\hat{s}}(u) = 0, \tag{2.8}$$

for all $u$.

Because $\{W_{\hat{s}}(u)\}$ is parallel along the geodesic $\eta_{\hat{s}}$, by (2.8) we know that the vector field $\{W_{\hat{s}}(u)\}$ is parallel Jacobian field along the geodesic $\eta_{\hat{s}}$ as well.

It is well-known that any variation vector field of 1-family of geodesics is a Jacobian field. By our construction, the variation field $\frac{\partial \phi_1}{\partial s}(\hat{s}, u)$ is also a Jacobi field along the geodesic $\eta_{\hat{s}}$. Moreover, $J_{\hat{s}}(u) = \frac{\partial \phi_1}{\partial s}(\hat{s}, u)$ has the same initial conditions as $W(u)$ does. Namely, we have $W_{\hat{s}}(0) = \frac{\partial \phi_1}{\partial s}(\hat{s}, 0) = c'_0(\hat{s})$ and $W'_{\hat{s}}(0) = 0 = \nabla_{\eta'_{\hat{s}}(0)} \frac{\partial \phi_1}{\partial s}(\hat{s}, 0)$. Therefore, by the uniqueness of Jacobi field with the given initial conditions, we arrive at $W_{\hat{s}}(u) = \frac{\partial \phi_1}{\partial s}(\hat{s}, u)$.

Consequently, $\{\frac{\partial \phi_1}{\partial s}(\hat{s}, u)\}$ is *a parallel Jacobian field* along each geodesic $\eta_{\hat{s}}$ for any $\hat{s} \in \mathbb{R}$. This completes the proof of Claim 2.3.

By induction on $j$, we let $c_{j-1}(s) = \phi_{j-1}(s, u_{j-1})$ for $j \geq 2$ and let $\{V_j(s)\}$ be a parallel vector field along the geodesic $c_{j-1}$ with initial condition $V_j(0) = \sigma'_j(0)$. Finally, we set

$$\begin{aligned} \phi_j : \quad & \mathbb{R} \times [0, u_j] \to M^n \\ & (s, u) \to Exp_{c_{j-1}(s)}[uV_j(s)]. \end{aligned} \tag{2.2.j}$$

By the proof of Claim 2.3, we con show that each $\phi_j$ is a totally geodesic isometric immersion. This completes the proof of Proposition 2.2 $\square$

### 2.c. Proofs of Theorem 2.1 and Theorem C

We begin an elementary example of non-smooth convex domains in $\mathbb{R}^2$ and in the unit sphere $S^2(1)$. Such an example inspires us to derive a monotone principle stated in Theorem C.

**Example 2.4.** We first consider the case of constant curvature.

(2.4.1) Let $\partial\Omega_1$ be an equilateral triangle in $\mathbb{R}^2$ such that each edge of $\partial\Omega_1$ has length $2\sqrt{3}$. Thus, the maximum inner scribed circle in $\Omega_1$ has radius 1, (i.e., $r_{max} = \sup_{x\in\Omega_1}\{d(x, \partial\Omega)\} = 1$). Let $\Omega_1$ be the solid triangle bounded by the triangle $\partial\Omega_1$ and let $\Omega_t = \{x \in \Omega | d(x, \partial\Omega_1) \geq 1 - t\}$ for $t \in [0, 1]$. Then $\Omega_0$ is a point.

(2.4.2) Similarly, we consider an equilateral geodesic triangle $\partial\hat{\Omega}$ in the unit sphere $S^2(1)$ such that each edge of $\partial\hat{\Omega}$ has length $\frac{\pi}{16}$. We also let $\hat{\Omega}$ be the solid triangle bounded by the triangle $\partial\hat{\Omega}$ in a hemisphere. Suppose that $r_0 = \sup_{x\in\Omega_{r_0}}\{d(x, \partial\hat{\Omega})\}$ and that $\Omega_t = \{x \in \Omega | d(x, \partial\hat{\Omega}) \geq r_0 - t\}$ for $t \in [0, r_0]$. Let $\{\theta_1(t), \theta_2(t), \theta_3(t)\}$ be the inner angles at three vertices of $\partial\Omega_t$. It follows from the Gauss-Bonnet Theorem that $\int_{\Omega_t} KdA + \sum_{j=1}^{3}[\pi - \theta_j(t)] = 2\pi$. Thus, we conclude that

$$\sum_{j=1}^{3}\theta_j(t) = \pi + \int_{\Omega_t} KdA, \tag{2.4.3}$$

where $K = 1$ is the sectional curvature of $S^(1)$. As $t \to 0^+$, the integral $\int_{\Omega_t} KdA$ gets smaller and smaller. Hence, the normal cones $\mathcal{N}^+(\Omega_t, \partial\Omega_t)$ is strictly increasing as $t \searrow 0^+$. $\square$

We would like to say a few words about the nearest point projection maps, because they have been frequently used in our paper.

**Definition 2.5.** (The reach and/or focal radius of a subset, Federer [Fe]) Let $A$ be a subset in a complete Riemannian manifold $(M^n, g)$. The reach of a subset $A$ of $M^n$ is the largest $\varepsilon$ such that if $x \in M^n$ and if $d(x, A) < \varepsilon$, then $A$ contains a unique point, $\mathcal{P}_A(x)$, nearest to $x$. In other words, we let $U_\epsilon(A) = \{x \in M^n | d(x, A) \leq \epsilon\}$ and let

$$\epsilon_0(A) = \sup\{\epsilon| \text{ There is a nearest point projection } \mathcal{P} : U_\epsilon(A) \to A\}$$

be the reach of $A$.

We would like to estimate the reach of a convex subset $A$ in a Riemannian manifold with nonnegative curvature. Let us consider the following example. Let $A$ be a great circle in the unit sphere $S^n(1)$. Since $A$ is a closed geodesic, $A$ is a convex subset of $S^n(1)$. Clearly, reach of $A$ is equal to $\frac{\pi}{2}$.

**Proposition 2.6.** *(compare Lemma 2.4 of [ChG]) Let $A \subset \Omega_T$ be a connected, convex subset in a Riemannian manifold with nonnegative curvature, let $K_0 = \max\{K(x)|x \in \Omega_{T+1}\}$, $Inj_{(M^n,g)}(\Omega_T)$ and $\mathcal{S}$ be as above. Then the subset $A$ has positive reach (or focal radius) bounded below by*

$$\epsilon_0(A) \geq \epsilon_T = \frac{1}{4}\min\{Inj_{(M^n,g)}(\Omega_T), \frac{\pi}{\sqrt{K_0}}, 1\},$$

*where $\epsilon_T$ is independent of choices of convex subsets $A \subset \Omega_T$.*

*Proof.* We prove Proposition 2.6 by a contradiction argument. For each $q \in M^n$ with $d(q, A) < \epsilon_T$, suppose contrary, there were two distinct nearest points $\{p, w\} \subset A$ with $d(p, q) = d(w, q) = d(A, q)$. Then we consider a geodesic triangle $\triangle_{pqw}$ whose sides are length-minimizing geodesic segments. Since $\{p, w\}$ are nearest point and $A$ is convex, the inner angles of $\triangle_{pqw}$ at $\{p, w\}$ must be greater than or equal to $\frac{\pi}{2}$, by the first variational formula.

On other hand, it follows from a triangle comparison theorem in [Kl, p219] that the angles of triangle $\triangle_{pqw}$ at $p$ and $w$ must be strictly less than $\frac{\pi}{2}$. This is a contradiction. Therefore, we have $\epsilon_0(A) \geq \epsilon_T > 0$. $\square$

For any subsets $A$ and $B$ in $M^n$, we consider the Hausdorff distance $Hd(A, B) = \sup\{\epsilon|U_\epsilon(A) \supset B, U_\epsilon(B) = A\}$.

It was shown in [Yim1] that there is a constant $\alpha_T \geq 1$

$$Hd(\partial\Omega_{t_1}, \partial\Omega_{t_2}) \leq \alpha_T|t_1 - t_2|, \tag{2.9}$$

for all $t_1 \leq t_2 \leq T$.

Let us clarify our notations before the proof of Theorem 2.1.

**Definition 2.7.** Let $\Omega$ be a compact and convex domain with the relative boundary $\partial\Omega$ in a complete Riemannian manifold $(M^n, g)$.

(2.7.1) (Inward Tangent Cone) For each boundary point $p \in \partial\Omega$, the inward tangent cone of $\Omega$ at $p$ is defined to be $T_p^-\Omega = \{v \in T_p(M^n)|Exp_p(tv) \in \Omega$ for some small $t > 0\}$, where $Exp_P$ is the exponential map of $(M^n, g)$ at $p$.

(2.7.2) (The whole linear tangent space) The linear space spanned by $T_p^-\Omega$ is called the whole linear tangent space of $\Omega$ at $p$, which is denoted by $T_p(\Omega)$;

**Proof of Theorem 2.1.**

We will use Proposition 2.6 to identify the outward normal cone $\mathcal{N}^+(\Omega_{t_0}, \partial\Omega_{t_0})$ with another cone as follows. Let $W = \mathcal{P}^{-1}(Q_{t_0})$. It follows from Proposition 2.3 that $V = T^-_{Q_{t_0}}(W) = \mathcal{N}^+_{Q_{t_0}}(\Omega_{t_0}, \partial\Omega_{t_0})$. Since $\Omega_{t_0}$ is convex, one can verify that $\mathcal{N}^+_{Q_{t_0}}(\Omega_{t_0}, \partial\Omega_{t_0})$ is a convex cone in $T_{Q_{t_0}}(M^n)$. Because $V = N^+_{Q_{t_0}}(\Omega_{t_0}, \partial\Omega_{t_0})$ is convex, we consider its relative boundary $\partial V$. For each unit outward normal vector $v_0 \in \partial V$, we let $\psi(t) = Exp_{Q_{t_0}}[(t - t_0)v_0]$. Then $\psi : [t_0, t_1] \to \Omega_{t_1}$ is a length minimizing geodesic from $Q_{t_0}$ to $\partial\Omega_{t_1}$.

Let $p_1 = \psi(t_1)$. For any $Q_{t_1} \in W \cap \partial\Omega_{t_1}$ with $Q_{t_1} \neq p_1$, we consider geodesic triangle $\triangle_{Q_{t_0}, Q_{t_1}, p_1}$. Since $\Omega_{t_1}$ is convex, we see that the inner angle of $\triangle_{Q_{t_0}, Q_{t_1}, p_1}$ at $p_1$ is $\leq \frac{\pi}{2}$, i.e., $\angle_{p_1}(Q_{t_1}, Q_{t_0}) \leq \frac{\pi}{2}$

On the other hand, the sectional curvature is nonnegative, by the classical angle comparison theorem of Alexandroff and Toponogov for $K \geq 0$, we know that the sum of inner angles of $\triangle_{Q_{t_0}, Q_{t_1}, p_1}$ is greater than or equal to $\pi$, (e.g., see [Kl, p220] or [Cha, p329]). It follows that

$$\angle_{Q_{t_1}}(p_1, Q_{t_0}) + \angle_{Q_{t_0}}(p_1, Q_{t_1}) \geq \pi - \angle_{p_1}(Q_{t_1}, Q_{t_0}) \geq \frac{\pi}{2}. \tag{2.10}$$

Let us now take a parallel transport $\mathbb{P}_\sigma$ where $\sigma : [0, \ell] \to \Omega_{a_j}$ is geodesic segment of unit speed from $Q_{t_0}$ to $Q_{t_1}$. We consider the Fermi coordinate system around the geodesic $\sigma$. Namely, we choose an ortho-normal frame $\{e_1, e_2, ..., e_m\}$ of $T_{Q_{t_0}}(\Omega_{a_j})$ such that $e_m = \sigma'(0)$. Suppose that $\{\vec{E}_1(t), ..., \vec{E}_m(t)\}$ is a parallel transport of $\{e_1, e_2, ..., e_m\}$ along $\sigma$. The Fermi coordinate system is given by the following map:

$$F: \quad \mathbb{R}^{m-1} \times [0, \ell] \to M^n$$

$$(x_1, ..., x_{m-1}, x_m) \to Exp_{\sigma(x_m)}[\sum_{k=1}^{m-1} x_k \vec{E}_k(s)]$$

It is well-known that the derivative of $F$ at the zero section is equal to identity, i.e.,

$$F_*|_{(0,...,0,x_m)} = id \tag{2.11}$$

for all $x_m$. Let $\mathbb{R}^2_0 \subset T_{Q_{t_0}}(M^n)$ be a 2-dimensional subspace spanned by $\sigma'(\ell)$ and the vector $\Psi'(t_0)$. Similarly, if $\phi : [0, t_1 - t_0] \to M^n$ be a geodesic of unit speed from $Q_{t_1}$ to $p_1$, then we consider a 2-plane $\mathbb{R}^2_\ell \subset T_{Q_{t_1}}(M^n)$ spanned by $\sigma'(\ell)$ and $\phi'(0)$. Then by (2.11) we have

$$\angle_{Q_{t_1}}(\mathbb{R}^2_\ell, \mathbb{P}_\sigma[\mathbb{R}^2_0]) = O(\ell^2), \tag{2.12}$$

where $O(\ell^2)$ denotes a term of order 2 in $\ell$. We also have $\ell = d(Q_{t_0}, Q_{t_1}) = O(|t_1 - t_0|)$ by (2.9), Lemma 1.1 of [Yim1] and its proof.

We consider $\vec{v}(t) = \mathbb{P}_\sigma(\vec{v}_0)$ at $\sigma(t)$ and an angle function

$$\beta_v(t) = \angle_{\sigma(t)}(\vec{v}(t), T^-_{\sigma(t)}(\Omega_t)).$$

It follows from (2.9)-(2.10) and (2.12) that

$$\begin{aligned}\beta_v(t_1) \le & \angle_{Q_{t_1}}(\vec{v}(t_1), p_1)\\ = & \pi - [\angle_{Q_{t_1}}(\sigma'(t_1), \vec{v}(t_1)) + \angle_{Q_{t_1}}(p_1, -\sigma'(t_1))] + O(\ell^2)\\ = & \pi - [\angle_{Q_{t_0}}(Q_{t_1}, p_1) + \angle_{Q_{t_1}}(p_1, Q_{t_0})] + O(\ell^2)\\ \le & \frac{\pi}{2} + O(\ell^2) = \frac{\pi}{2} + O(|t_1 - t_0|^2).\end{aligned} \tag{2.13}$$

Recall that $\vec{v}(t_0) = v_0$ is an outer normal vector of $\Omega_{t_0}$. Thus, $\beta_v(t_0) = \frac{\pi}{2}$. It follows from (2.13) that the one-sided derivative of angle function $\beta_v$ is non-positive, i.e.,

$$\frac{\partial^+(\beta_v)}{\partial t}|_{t_0} \le 0, \tag{2.14}$$

for all $v \in \partial V$. Theorem 2.1 is a direct consequence of (2.14). $\square$

In order to prove Theorem C using Theorem 2.1, we approximate the trajectories of (0.3) by the broken geodesics as follows.

Let $\Lambda_k = \{a_{j-1} = t_0 < t_1 < ... < t_{2^k} = T\}$ be the partition of $[a_{j-1}, T]$ of mesh size $\le 4\frac{T-a_{j-1}}{k}$. Let $\mathcal{P}_i = \mathcal{P}_{j,k,i} : \Omega_{t_i} \to \Omega_{t_{i-1}}$ be the nearest point projection. By Proposition 2.6, each convex domain $\Omega_{t_i}$ has positive reach (focal radius). If the mesh size of the partition $\Lambda_k$ is sufficiently small (less than $\epsilon_T$ given by (2.3) ), then the nearest point projection $\mathcal{P}_i = \mathcal{P}_{i,j,k}$ is well-defined. For each point $Q \in \partial\Omega_T$, we let $Q_0 = Q$ and $Q_i = \mathcal{P}_i(Q_{i-1})$. Let $\sigma_{k,Q}$ be the broken geodesic joining $\{Q = Q_0, ...., Q_{2^k}\}$. Then Sharafutdinov and Yim obtained the following:

**Proposition 2.8.** *([Sh, p563], [Yim1-2]) Let $T > a_{j-1}, \Lambda_k, \epsilon_T$, $Q \in b\Omega_T$, $\{Q_0, Q_1, ..., Q_k\}$ and $\sigma_{k,Q}$ be as above. Then, as $k \to +\infty$, the broken geodesic $\sigma_{k,Q}$ will converges to the trajectory $\sigma_{\infty,Q}$ of the semi-flow (0.3) with initial point $Q$ (in the Lipschitz topology).*

Theorem C follows from Theorem 2.1 and proposition 2.8.

## 3. Distance non-increasing retractions compatible with the vertical Fermi maps

In this section, we study the smoothness of distance non-increasing retraction $\Psi : M^n \to \mathcal{S}$ from the manifold $M^n$ of nonnegative curvature to its soul $\mathcal{S}$. The main technique is to show that such a retraction $\Psi$ is compatible with various smooth Fermi maps which we now describe.

**Definition 3.1.** Let $A$ be a subset of a complete smooth Riemannian manifold $M^n$. Suppose that $A$ has positive reach $\epsilon_0(A) > 0$.

(3.1.1) The outward normal cone of $A$ in $M^n$ is defined to be

$$\mathcal{N}^+(A, M^n) = \{(p, v) | p \in A, d(Exp_p(tv), A) = t \text{ for some } t > 0, \text{ or } v = 0\}.$$

(3.1.2) The Fermi map along the subset $A$ is defined to be

$$\begin{aligned}\mathcal{F}_A:\ \ & \mathcal{N}^+(A, M^n) \to M^n\\ & (p, v) \to Exp_p(v)\end{aligned}$$

(3.1.3) A retraction $\Psi : M^n \to \mathcal{S}$ is said to be compatible with $\mathcal{F}_A$ if $\Psi(\mathcal{F}_A(p, v)) = \Psi(p)$ for all $(p, v) \in \mathcal{N}^+(A, M^n)$.

For our application in this paper, we consider the so-called the Sharafutdinov retraction $\phi : M^n \to \mathcal{S}$ from an open manifold $M^n$ to its soul. The Sharafutdinov retraction is constructed by the piecewise flow (0.3).

**Proposition 3.2.** *([Sh], [Yim1]) The Sharafutdinov retraction $\phi : M^n \to \mathcal{S}$ given by the piece-wise semi-flow (0.3) is a distance non-increasing retraction.*

Proof of Proposition 3.2 only uses the convexity of the exhaustion $\{\Omega_t\}$. The curvature assumption of $K \geq 0$ does not play any role in the proof of Proposition 3.2.

In presence of nonnegative sectional curvature, Perelman [Per] showed that *any* distance non-increasing retraction $\Psi : M^n \to \mathcal{S}$ from $M^n$ to its soul is compatible with the Fermi map along the soul. Perelman's result was improved by Guijarro [Gu]. In both Perelman and Guijarro's approach, Berger comparison theorem plays an important role. In fact, Berger comparison theorem implies that if $K \geq 0$, then the Fermi map $\mathcal{F}_{\mathcal{S}}$ along the soul is distance non-increasing in "horizontal directions" as well. In other words, the evolution of "horizontal directions" starting from soul is length non-increasing. This observation together with Proposition 3.2 gives rise to Perelman's rigidity theorem [Per] about the uniqueness of the retraction $M^n \to \mathcal{S}$.

For the completeness of our paper, we give a short proof of Berger Comparison.

**Proposition 3.3.** *(Berger Comparison for $K \geq 0$) Suppose that $\gamma : [a, b] \to M^n$ is a geodesic of unit speed and $\{V(s)\}$ is a parallel unit normal vector field along $\gamma$. Let $\gamma_t : [a, b] \to M^n$ be the horizontal curve given by*

$$\gamma_t(s) = \phi(s, t) = Exp_{\gamma(s)}[t\vec{V}(s)]. \tag{3.1}$$

*Then, for any pair of $t_1 < t_2$, the length $L(\gamma_{t_2})$ of $\gamma_{t_2}$ is always less than or equal to the length of $\gamma_{t_1}$, i.e.,*

$$L(\gamma_{t_2}) \leq L(\gamma_{t_1}). \tag{3.2}$$

*Equality holds in (3.2) if and only if $\phi([a, b] \times [t_1, t_2])$ is a totally geodesic immersed flat strip.*

*Proof.* For any given $\hat{s} \in [a, b]$, let us consider a hypersurface

$$\Sigma^{n-1}_\epsilon(\gamma(\hat{s})) = \{Exp_{\gamma(\hat{s})}w | w \perp V(\hat{s}), |w| \leq \epsilon\}. \tag{3.3}$$

By the definition of the exponential map, we see that the hypersurface $\Sigma^{n-1}_\epsilon(\gamma(\hat{s}))$ is totally geodesic at $\gamma(\hat{s})$. Thus, the second fundamental form of $\Sigma^{n-1}_\epsilon(\gamma(\hat{s}))$ at at $\gamma(\hat{s})$ is zero.

Let $r_{\hat{s}}(x) = d(x, \Sigma^{n-1}_\epsilon(\gamma(\hat{s})))$ and $J_{\hat{s}}(t) = \frac{\partial \phi}{\partial s}(\hat{s}, t)$. A computation shows that

$$\frac{d[\log |J_{\hat{s}}(t)|]}{dt} = \mathrm{Hess}(r_{\hat{s}})(\frac{J_{\hat{s}}(t)}{|J_{\hat{s}}(t)|}, \frac{J_{\hat{s}}(t)}{|J_{\hat{s}}(t)|}) = \langle \mathcal{B}\frac{J_{\hat{s}}(t)}{|J_{\hat{s}}(t)|}, \frac{J_{\hat{s}}(t)}{|J_{\hat{s}}(t)|}\rangle. \tag{3.4}$$

Since $K \geq 0$, using Proposition B for $\mathcal{B}(t) = \mathrm{Hess}(r_{\hat{s}})|_{\sigma_{\hat{s}}(t)}$ and the initial condition $\mathcal{B}(0) = 0$, we see that $\mathcal{B}(t) \leq 0$ for $t \geq 0$ and

$$\frac{d[\log |J_{\hat{s}}(t)|]}{dt} \leq 0 \tag{3.5}$$

for all $t \geq 0$. Thus, it follows from (3.5) that (3.2) holds. Moreover, if equality holds in (3.2), then $\{J_{\hat{s}}(t)\}$ is a parallel Jacobi field along the geodesic $\sigma_{\hat{s}} : t \to \phi(\hat{s}, t)$ for all $(\hat{s}, t) \in [a, b] \times [t_1, t_2]$. It follows that if equality holds in (3.2), then $\phi$ is a totally geodesic isometric immersion. $\square$

**Proposition 3.4.** *(Perelman [Per]) Let $M^n$ be a complete open $C^\infty$-smooth manifold with nonnegative sectional curvature and let $\mathcal{S}$ be its soul. Suppose that $\Psi : M^n \to \mathcal{S}$ be any distance non-increasing retraction. Then*

*(3.4.1) $\Psi$ must be compatible with the Fermi map $F_{\mathcal{S}} : \mathcal{N}^+(\mathcal{S}, M^n) \to M^n$;*

*(3.4.2) For each $(p, \vec{v}) \in \mathcal{N}^+(\mathcal{S}, M^n)$ and $\vec{w} \in T_p(\mathcal{S})$ with $\vec{v} \neq 0 \neq \vec{w}$, the surface $\Sigma_{\{\vec{w}, \vec{v}\}} = Exp_p[span_{\mathbb{R}}\{v, w\}]$ is a totally geodesic flat immersed Euclidean plane.*

*Consequently, the map $\Psi$ must be $C^\infty$-smooth at the regular values of the Fermi map $\mathcal{F}_{\mathcal{S}}$. The distance non-increasing retraction $\Psi : M^n \to \mathcal{S}$ is $C^\infty$-smooth almost everywhere.*

*Proof.* We reproduce Perelman's proof here because we need to use it somewhere else in our paper.

Let $\tilde{U}^N_t(\mathcal{S}) = \{(p, \vec{v}) \in \mathcal{N}^+(\mathcal{S}, M^n) | |\vec{v}| \leq t, p \in \mathcal{S}\}$. We consider the distance function

$$\eta(t) = \max\{d(p, \Psi[\mathcal{F}(p, \vec{u})]) | |\vec{u}| = t, p \in \mathcal{S}\}.$$

It is sufficient to verify the following.

**Claim 3.5.T.** Let $M^n$, $\mathcal{S}$ and $\eta$ be as above. Then

(3.5.1.T). The left derivative of $\eta$ is non-positive:

$$\frac{d\eta}{d^- t}(t) \leq 0 \quad \text{and} \quad \eta(t) = 0 \tag{3.6}$$

for all $t \in [0, T]$;

(3.5.2.T). For any geodesic segment of unit speed $\gamma : [a,b] \to \mathcal{S}$ and any vector field $\vec{V} \in \partial\tilde{U}_1^N(\mathcal{S})$ parallel along $\gamma$, then the map

$$\begin{aligned} \phi : \quad & [a,b] \times [0,T] \to M^n \\ & (s,t) \to Exp_{\gamma(s)}[t\vec{V}(s)] \end{aligned} \tag{3.7}$$

is totally geodesic isometric immersion.

The proof of Claim 3.5.T is based on a bootstrap argument as in the proof of Proposition 2.0 with some modifications. We first verify that (3.5.1.T)-(3.5.2.T) hold for $T = \epsilon_0$, where $\epsilon_0 = \frac{1}{4}\mathrm{Inj}(\mathcal{S})$ and $\mathrm{Inj}(\mathcal{S})$ is the injectivity radius of $\mathcal{S}$. Afterwards, we verify that (3.5.1.T)-(3.5.2.T) hold for $T = j\epsilon_0$ with $j = 2, 3, ....$

Fix any $\hat{s} \in [a,b]$, the variation field $J_{\hat{s}}(t) = \frac{\partial\phi}{\partial s}(\hat{s},t)$ is a Jacobi field along the vertical geodesic $\sigma_{\hat{s}} : t \to \sigma_{\hat{s}}(t) = \phi(\hat{s},t)$. We now use Berger comparison theorem (Proposition 3.3) to prove (3.5.1.T) for $T = \epsilon_0$.

For each $t_0 \in (0,\epsilon_0]$, since $\mathcal{S}$ is compact, there is a $(x_0, v_0) \in \partial[\tilde{U}_1^N(\mathcal{S})]$ such that $\eta(t_0) = d(Exp_{x_0}(t_0v_0), \Psi[Exp_{x_0}(t_0v_0)])$.

Let $\bar{x}_0 = \Psi[Exp_{x_0}(t_0v_0)]$ and $\hat{\gamma} : [-\eta(t_0), 0] \to \mathcal{S}$ be a geodesic of unit speed from $\bar{x}_0$ to $x_0$ such that $\hat{\gamma}(-\eta(t_0)) = \bar{x}_0$ and $\hat{\gamma}(0) = x_0$. Let $\{V(s)\}$ be a parallel vector field along $\hat{\gamma}$ with $V(0) = v_0$. Similarly, we define

$$\begin{aligned} \hat{\phi} : \quad & [0,\epsilon_0] \times [0,t_0] \to M^n \\ & (s,t) \to Exp_{\hat{\gamma}(s)}[t\vec{V}(s)]. \end{aligned} \tag{3.8}$$

Let $\hat{\gamma}_t : [0,\epsilon_0] \to M^n$ be given by $\hat{\gamma}_t(s) = \hat{\phi}(s,t)$. We also let $x_1 = \hat{\gamma}(\epsilon_0)$, $\bar{x}_1 = \Psi[Exp_{x_1}(t_0V)]$. By the distance non-increasing property of $\Psi$ and Proposition 3.3, we have

$$d(\bar{x}_0, \bar{x}_1) \le L(\hat{\gamma}_{t_0}) \le L(\hat{\gamma}) = d(x_0, x_1). \tag{3.9}$$

We now show that equality holds in (3.9) as follows. By our choice of $t_0v_0$, we have

$$d(x_0, \bar{x}_0) \ge d(x_1, \bar{x}_1) \tag{3.10}$$

Since $x_0$ lies between $\bar{x}_0$ and $x_1$ on the geodesic $\hat{\gamma}$, we have

$$d(x_0, \bar{x}_0) + d(x_0, x_1) = d(\bar{x}_0, x_1) \le d(x_1, \bar{x}_1) + d(\bar{x}_0, \bar{x}_1) \tag{3.11}$$

Subtracting (3.10) from (3.11), we obtain the reversed inequality

$$d(x_0, x_1) \le d(\bar{x}_0, \bar{x}_1). \tag{3.12}$$

Hence, it follows (3.9) and (3.12) that

$$d(x_0, x_1) = d(\bar{x}_0, \bar{x}_1) \text{ and } L(\hat{\gamma}_{t_0}) = L(\hat{\gamma}). \tag{3.13}$$

Combining (3.13) and Proposition 3.3, we see that

$$\hat{\phi} \text{ is a totally geodesic isometric immersion.} \tag{3.14}$$

As before, we let $\sigma_s(t) = \hat{\phi}(s,t)$. As $\delta \searrow 0$, by (3.14) we have

$$[d(\sigma_{\epsilon_0}(t_0-\delta), \sigma_0(t_0))]^2 = [d(\sigma_{\epsilon_0}(t_0), \sigma_0(t_0))]^2 + \delta^2. \tag{3.15}$$

Let $O(\delta^2)$ denote a term of order 2 in $\delta$. Finally, by (3.14)-(3.15) and the distance non-increasing property of $\Psi$, Perelman observed

$$\begin{aligned}
\eta(t_0-\delta) \quad & \geq d\big(x_1, \Psi[\sigma_{\epsilon_0}(t_0-\delta)]\big) \\
& \geq d(x_1, \bar{x}_0) - d(\bar{x}_0, \Psi[\sigma_{\epsilon_0}(t_0-\delta)]) \\
& \geq d(\bar{x}_0, x_1) - d(\sigma_0(t_0), \sigma_{\epsilon_0}(t_0-\delta)) \\
& = d(\bar{x}_0, x_1) - [d(\sigma_0(t_0), \sigma_{\epsilon_0}(t_0)) + O(\delta^2)] \\
& = d(\bar{x}_0, x_1) - [d(x_0, x_1) + O(\delta^2)] \\
& = d(\bar{x}_0, x_0) - O(\delta^2) = \eta(t_0) - O(\delta^2).
\end{aligned} \tag{3.16}$$

This shows that $\frac{d\eta}{d^- t}(t_0) \leq 0$ for all $t_0 \in [0, \epsilon_0]$.

Since $\eta(0) = 0$ and $\eta(t) \geq 0$, one concludes that $\eta(t) = 0$ for all $t_0 \in [0, \epsilon_0]$. Thus, we showed that

$$\Psi[\mathcal{F}_{\mathcal{S}}(p, \vec{u})] = p, \tag{3.17}$$

holds for all $\|\vec{u}\| \leq \epsilon_0$. Using (3.17), the distance non-increasing property of $\Psi$ and Proposition 3.3 again, we obtain (3.5.2.T) holds for $T = \epsilon_0$.

We now using a bootstrap argument and induction on j to verify Claim 3.5.T holds for $T = T_j = j\epsilon_0$ with $j = 2, 3, ....$ This completes the proof of Proposition 3.4. □

### 3.a. A sufficient condition for the smoothness of $\Psi$.

By Proposition 3.4, we see that any distance non-increasing retraction $\Psi : M^n \to \mathcal{S}$ is $C^\infty$-smooth away from the focal-loci of the soul $\mathcal{S}$ in $M^n$. We are going to elaborate this observation to derive a sufficient condition for the $C^\infty$-smoothness of $\Psi$ everywhere.

**Proposition 3.6.** *Let $\Psi : M^n \to \mathcal{S}$ be as in Proposition 3.4. Suppose that for each $p_0 \in M^n$, there exist a $C^\infty$-smooth submanifold $A(p_0) \subset M^n$ and a Fermi map $\mathcal{F}_{A(p_0)} : \mathcal{N}^+(A(p_0), M^n) \to M^n$ such that*

*(1) $\mathcal{F}_{A(p_0)}$ is compatible with $\Psi$;*

*(2) $p_0 \in A(p_0)$;*

*(3) $\Sigma_q = A(p_0) \cap \Psi^{-1}(q)$ is $C^\infty$-smooth for each $q \in \mathcal{S}$.*

*(4) $Hd(\Sigma_q, \Sigma_{q'}) = d(q, q')$ for any pair $\{q, q'\} \subset \mathcal{S}$; The family $\{\Sigma_q\}$ is $C^\infty$-smooth in $q \in \mathcal{S}$.*

*Then the map $\Psi$ is $C^\infty$-smooth everywhere.*

*Proof.* Notice that, for any $C^1$-smooth Fermi-map, the derivative $D\mathcal{F}$ of the Fermi map $\mathcal{F}$ satisfies:

$$D\mathcal{F}_{A(p_0)}|_{(p,0)} = id.$$

Hence, $\mathcal{F}_{A(p_0)}$ is a local diffeomorphism along the zero-section of $\mathcal{N}^+(A(p_0), M^n)$. By our assumptions (1)-(4), there is a compatible Fermi-coordinates around $p_0$:

$$G_{p_0} : (x_1, ..., x_k; y_1, ..., y_l; \vec{u}) \to Exp_{\phi(x_1,...,x_k;y_1,...,y_l)}\vec{u}$$

such that $\Psi[G_{p_0}(x_1, ..., x_k; y_1, ..., y_l; \vec{u})] = (x_1, ..., x_k)$ for $(x_1, ..., x_k)$ in a small neighborhood of $\Psi(p_0)$. By our assumption (4), $\phi(x, y)$ is a $C^\infty$-smooth in $x$ and $y$. Furthermore, $\phi$ is an immersion. It follows that our map $\Psi$ is $C^\infty$-smooth around $p_0$. $\square$

Inspired by Proposition 3.6, we need to construct a compatible Fermi-map $\mathcal{F}_{A(p_0)}$ and a $C^\infty$ submanifold $A(p_0)$ for any $p_0$ in the next two subsections.

### 3.b. Horizontal Fermi diffeomorphisms between fibres.

In order to construct compatible Fermi maps $\{\mathcal{F}_{A(p_0)}\}$ with different base submanifold $\{A(p_0)\}$, we need to recall a result of L. Guijarro [Gu].

**Definition 3.7.** A map $\Psi : M^n \to \mathcal{S}$ between two Riemannian manifolds is called a submetric if $\Psi$ maps a ball $B_r(x)$ of radius r centered at $x$ onto $B_r(\Psi(x))$, i.e.; $B_r(\Psi(x)) = \Psi(B_r(x))$ for all $x \in M^n$ and $r \geq 0$.

For each $p \in M^n$, the tangent subspace $\mathcal{V}_p = T_p[\Psi^{-1}(\Psi(p))]$ is called the vertical subspace associated with $\Psi$. The orthogonal complement of $\mathcal{V}_p$ is denoted by $\mathcal{H}_p$. The distribution $\{\mathcal{H}_p\}_{p\in M^n}$ is called the horizontal distribution.

In general, the horizontal distribution $\{\mathcal{H}_p\}_{p\in M^n}$ is not integrable. This leads the main difficulty in our construction of $\{A(p_0)\}$ described in Proposition 3.6.

For any distance non-increasing retraction $\Psi : M^n \to \mathcal{S}$, in the presence of nonnegative sectional curvature, $\Psi$ must be a submetric map. In fact, by Perelman's flat strip theorem (Proposition 3.4 above) the fibres $\{\Psi^{-1}(q)\}_{q\in\mathcal{S}}$ form an equidistant family of submanifolds. Namely, we have

$$Hd(\Psi^{-1}(q), \Psi^{-1}(q')) = d(q, q') \tag{3.18}$$

holds for any pair $\{q', q\} \subset \mathcal{S}$.

**Proposition 3.8.** *(compare [BG]) Let $\Psi : M^n \to \mathcal{S}$ be a submetric map. Then each fibre $F_q = \Psi^{-1}(q)$ with $q \in \mathcal{S}$ has positive reach $\geq \delta$, where $\delta$ is the injectivity radius of the soul $\mathcal{S}$. Consequently, each fibre is a $C^{1,1}$-submanifold. Moreover the horizontal Fermi map $\mathcal{F}_{F_q} : \mathcal{N}^+(F_q, M^n) \to M^n$ is a local diffeomorphism around a tube of $F_q$.*

*Proof.* Let $A \subset \mathbb{R}^n$ be a subset with positive reach $\delta_A > 0$. Then, by Theorem 5.9 of [Fe], the hypersurface $\partial U_s(A)$ is of $C^{1,1}$-smooth for $0 < s < \delta$, where $U_s(A) = \{x \in M^n | d(x, A) < s\}$.

For each $\vec{w} \in T_q(\mathcal{S})$, we have a horizontal lifting vector field $\{\vec{W}(p)\}_{p \in F_q}$. We are going to show that $W(p)$ is Lipschitz continuous in $p$, by the result of Federer above. We may assume that $w \neq 0$ and $\|w\| = 1$ after re-scaling if needed. Choose $q' = Exp_q[-sw]$ for $s = \frac{1}{4}\delta$, where $\delta$ is injectivity radius of $\mathcal{S}$. Setting $A = F_{q'}$, we observe that $F_q \subset \partial U_s(A)$. Furthermore, the outward unit vector field $\tilde{W}$ of $\partial U_s(A)$ satisfies

$$\tilde{W}|_{F_q} = W. \tag{3.19}$$

Federer already showed that $\tilde{W}$ is Lipschitz continuous on $\partial U_s(A)$. Hence, $W$ is Lipschitz continuous on $F_q$ by (3.19). □

Using Proposition 3.8 and its proof, Guijarro further investigated relations among fibres using horizontal Fermi maps.

**Proposition 3.9.** *([Gu]) Suppose that $M^n$ is an open manifold with nonnegative curvature and $\mathcal{S}$ is its soul. Let $\Psi : M^n \to \mathcal{S}$ be a Riemannian submetric map. Given any geodesic $\bar{\gamma} : [a, b] \to \mathcal{S}$ from $q$ to $q'$, there is a horizontal $C^1$-diffeomorphism $h_{\bar{\gamma}} : F_q \to F_{q'}$ obtained by lifting $\bar{\gamma}$ horizontally to each $p \in F_q$. Such a map $h_{\bar{\gamma}}$ coincides the nearest point projection from $F_q$ to $F_{q'}$, which is compatible with the horizontal Fermi map $\mathcal{F}_{F_{q'}}$ in a tubular neighborhood $U_s(F_{q'})$ of $F_{q'}$.*

Applying Proposition 3.9 to any closed broken geodesic $\bar{\gamma} : [0, 1] \to \mathcal{S}$ based at $q$, we get a corresponding self-diffeomorphism $h_{\bar{\gamma}} : F_q \to F_q$ where $\bar{\gamma}(0) = q = \bar{\gamma}(1)$. Since any Lipschitz curve $\bar{\alpha} : [0, 1] \to \mathcal{S}$ can be approximated by a sequence of broken geodesics, Guijarro considered the normal holonomy associated with horizontal Fermi maps.

**Definition 3.10.** Let $\Psi : M^n \to \mathcal{S}$ be a Riemannian submetric map. For each $q \in \mathcal{S}$, we let $\mathcal{G}^{\Psi}_{F_q} \subset \text{Diff}(F_q)$ be the sub-group of self-diffeomorphisms of $F_q$ generated by horizontal Fermi diffeomorphisms $h_{\bar{\alpha}}$ as above, where $\bar{\alpha}$ is a closed contractible loop based at $q$ in $\mathcal{S}$. The group $\mathcal{G}^{\Psi}_{F_q}$ is called horizontal holonomy group associated with $\Psi$ acting on the fibre $F_q = \Psi^{-1}(q)$.

In the presence of nonnegative sectional curvature, we will use the group orbits of $\mathcal{G}^{\Psi}_{F_q}$ to construct a family of $\{A(p_0)\}$ described in Proposition 3.6 above.

**3.c. Nonnegative curvature and compatible vertical Fermi maps.**

The main tool in construction of $\{A(p_0)\}$ and *compatible vertical* Fermi maps is the following.

**Proposition 3.11.** *([Gu]) Let $\Psi : M^n \to \mathcal{S}$ be as in Proposition 3.4. Suppose that $\vec{v} \in \mathcal{V}_p$ is a vertical unit vector that stays vertical under parallel transport along any horizontal piecewise broken geodesic. Then, for any horizontal geodesic $\alpha : \mathbb{R} \to M^n$ with $\alpha(0) = p$ of unit speed and the parallel vector field $\{V(s)\}$ along $\alpha$ with $V(0) = v$, we have*

*(1) $\Psi\big(Exp_{\alpha(s)}[tV(s)]\big) = \Psi(\alpha(s))$ for all $s, t$;*

*(2) The map*

$$\begin{aligned} \phi :\ & \mathbb{R} \times \mathbb{R} \to M^n \\ & (s,t) \to Exp_{\alpha(s)}[tV(s)] \end{aligned}$$

*is a totally geodesic isometric immersion;*

We are now in the final step of our construction of compatible Fermi map at each point $p_0 \in M^n$.

**Proof of Theorem E.**

For each $p_0 \in M^n$, we consider $q_0 = \Psi(p_0)$ and the fibre $F_{q_0} = \Psi^{-1}(q_0)$. Let us choose $A(p_0) = \{p_1 \in M^n | p_1 = h_{\bar{\alpha}}(p_0), \bar{\alpha}(0) = \Psi(p_0), \bar{\alpha}(1) = \Psi(p_1), \bar{\alpha} : [0,1] \to \mathcal{S}$ is Lipschitz continuous $\}$.

By Proposition 3.6, it is sufficient to verify the following:

**Claim 3.12** Let $\Psi : M^n \to \mathcal{S}$ and $A(p_0)$ be as above. Then

(3.12.1) For each $q \in \mathcal{S}$, the intersection $\Sigma_q = A(p_0) \cap F_q$ is a $C^\infty$ submanifold;

(3.12.2) For each $(p, \vec{v}) \in \mathcal{N}^+(A(p_0), M^n)$, the geodesic $\sigma_{(p,\vec{v})}(t) = Exp_p(t\vec{v})$ lies entirely in $F_{\Psi(p)}$.

For (3.12.1), we observe that the group $\mathcal{G}^\Psi_{F_q}$ is independent of choice of $p \in F_q$. Let $\delta$ be the focal radius $\mathcal{S}$ in $M^n$. Then the Fermi map $\mathcal{F}_\mathcal{S}$ is a local diffeomorphism in a small tubular neighborhood of $\mathcal{S}$. Let $\mathrm{Hol}^N_q$ be the holonomy group of the linear vector bundle $\mathcal{N}^+(\mathcal{S}, M^n)$ along $\mathcal{S}$ over the contractible loops based at $q$. If $k = n - dim(\mathcal{S})$ is the codimension of the soul $\mathcal{S}$, then it is well-known that $\mathrm{Hol}^N_q$ is a sub-group of the orthogonal group $SO(k)$, see [KN]. Since $\mathrm{Hol}^N_q$ is an analytic group and $\mathcal{G}^\Psi_{F_q}$ is conjugate to $\mathrm{Hol}^N_q$ via the local diffeomorphism $\mathcal{F}_\mathcal{S}$, the group $\mathcal{G}^\Psi_{F_q}$ must be a $C^\infty$-smooth Lie group as well, i.e., $\mathcal{G}^\Psi_{F_q} = \mathcal{F}_\mathcal{S} \circ [\mathrm{Hol}^N_q] \circ [\mathcal{F}^{-1}_\mathcal{S}|_{U_\epsilon(\mathcal{S})}]$, where $U_\epsilon(\mathcal{S})$ is the $\epsilon$-neighborhood of $\mathcal{S}$ in $M^n$.

For each $p \in \Sigma_q$, we let $I_p = \{h_{\bar{\alpha}} | h_{\bar{\alpha}}(p) = p\}$ be the isotropic subgroup of $\mathcal{G}^{\Psi}_{F_q}$ and let $\mathcal{O}_p$ be the orbit of $\mathcal{G}^{\Psi}_{F_q}$ which passes through $p$.

Then $\Sigma_q = \mathcal{O}_p = \mathcal{G}^{\Psi}_{F_q}/I_p$ for any $p \in \Sigma_q$. It follows that any orbit $\Sigma_q$ of $\mathcal{G}^{\Psi}_{F_q}$ is a $C^\infty$-smooth submanifold. This completes the proof of (3.12.1).

For (3.12.2), by Proposition 3.9, it is sufficient to verify the following:

**Assertion 3.13.** *For each $(p, \vec{v}) \in \mathcal{N}^+(A(p_0), M^n)$, the vector $\vec{v}$ stays vertical under parallel transportation along any horizontal piecewise geodesic.*

*Equivalently, the tangent subspace $T_p\big(A(p_0)\big)$ is invariant under parallel transportation along any horizontal piecewise geodesic.*

Assertion 3.13 has been established in Lemma 5.4 of [Gu] for the special case when $p$ is a focal point of the soul $\mathcal{S}$ and when $v$ is orthogonal to $\mathcal{F}_{\mathcal{S}}[\partial \tilde{U}^N_t(\mathcal{S})]$, where $t = |w|$ for some $(q, w) \in \mathcal{F}^{-1}_{\mathcal{S}}(p)$ and $\tilde{U}^N_t(\mathcal{S}) = \{(q, w) \in \mathcal{N}^+(\mathcal{S}, M^n) | |w| = t\}$.

In general, we consider the following integrable distribution

$$T_p(A(p_0)) = \mathcal{H}_p \oplus T_p(\mathcal{O}_p). \tag{3.20}$$

and

$$T_p(M^n) = T_p(A(p_0)) \oplus \big[T_p(A(p_0))\big]^\perp.$$

If $\xi \in T_p(M^n)$, we let $[\xi]^\perp$ be the component of $\xi$ in $\big[T_p(A(p_0))\big]^\perp$.

If $\alpha : [0, 1] \to M^n$ is a horizontal geodesic segment contained in $A(p_0)$, then $h_{\bar{\alpha}}(\mathcal{O}_p) = \mathcal{O}_{h_{\bar{\alpha}}(p)}$. Therefore, all members of the family $\{\mathcal{O}_p\}_{p \in A(p_0)}$ have the same dimension. If $Dh_{\bar{\alpha}}$ denotes the derivative of $h_{\bar{\alpha}}$, then we have $[Dh_{\bar{\alpha}}]\big(T_p(\mathcal{O}_p)\big) = T_{h_{\bar{\alpha}}(p)}[\mathcal{O}_{h_{\bar{\alpha}}(p)}]$.

Let us now consider a Jacobian field $\{J(s)\}$ along the horizontal geodesic $\alpha : [0, \infty) \to M^n$ with $\alpha(0) = p$ and $J(0) \in T_p(\mathcal{O}_p)$, we have shown that

$$J(s) \in T_{\alpha(s)}[\mathcal{O}_{\alpha(s)}] \subset T_{\alpha(s)}(A(p_0)) \tag{3.21}$$

for all $s$. We now claim that $J'(s) \in T_{\alpha(s)}(A(p_0))$. Otherwise, we let $s_0 = \max\{\hat{s} | [J'(s)]^\perp = 0 \text{ on } [0, \hat{s}]\}$. We would have $[J(s_0+\epsilon)]^\perp = \big[\int_{s_0}^{s_0+\epsilon} \mathbb{P}_\alpha(J'(s))ds\big]^\perp \neq 0$, which contradicts to (3.21).

Similarly, if $\{J(s)\}$ is a Jacobi field with $J(0) \in \mathcal{H}_p$, then by the definition of $A(p_0)$ we see that $J(s) \in T_{\alpha(s)}(A(p_0))$. For the same reason as above, we can show that $J'(s) \in T_{\alpha(s)}(A(p_0))$.

We now translate the above result in terms of the Hessian of distance functions. For each fibre $F_{q'} = \Psi^{-1}(q')$, we consider the distance function $r_{q'}(x) = d(x, F_{q'})$. We have shown the linear map

$$\begin{aligned} \mathcal{B}_{q'} : \quad & T_p(M^n) \to T_p(M^n) \\ & J \to \nabla_J(\nabla r_{q'}) = \nabla_{\nabla r_{q'}} J \end{aligned}$$

keeps the subspace $T_p(A(p_0))$ invariant, where $p \in F_q$ with $q \neq q'$ but $d(q, q')$ is sufficiently small. Clearly, the map $\mathcal{B}_{q'}$ is symmetric because

$$\langle \mathcal{B}_{q'}(\vec{v}), \vec{w} \rangle = \mathrm{Hess}(r_{q'})(\vec{v}, \vec{w}).$$

By the argument above, we have shown that

$$[J'(s)]^{\perp} = 0 \tag{3.22}$$

for any Jacobi field along a horizontal geodesic segment with $J(0) \in T_p(A(p_0))$.

Suppose that $m = \dim[T_p(A(p_0))]$. Let $\alpha : [0, \ell] \to M^n$ be a horizontal geodesic segment with $\sigma(0) = p$. If $\{E(s)\}$ is parallel vector field along $\alpha$ with $E(0) \in T_{\sigma(0)}(A(p_0))$, our goal is to show that $E(s) \in T_{\sigma(s)}(A(p_0))$ for all $s \in [0, \ell]$.

To do this, we consider $\hat{E}(s) = \sum_{i=1}^{m} x_i(s) J_i(s)$, where $\{J_1(s), ...., J_m(s)\}$ span $T_{\sigma(s)}(A(p_0))$. By (3.22), we may assume that

$$J_i'(s) = \sum_{i=1}^{m} b_{ik}(s) J_k(s),$$

for $i = 1, 2, ..., m$. Let $\hat{\mathcal{B}}(s) = (b_{ik}(s))$ be the $m \times m$ sub-matrix-valued function.

Thus, we derived an ODE sub-system

$$0 = \hat{E}'(s) = \sum_{i=1}^{m} [x_i'(s) J_i(s) + x_i(s) J_i'(s)] = \sum_{k=1}^{m} [x_k'(s) + \sum_{i=1}^{m} b_{ik} x_k(s)] J_k(s) \tag{3.23}$$

which is equivalent to

$$\frac{d\vec{x}}{ds}(s) = -\hat{\mathcal{B}}(s)\vec{x}(s). \tag{3.24}$$

The ODE sub-system (3.24) is always solvable. Clearly, any solution $\{\hat{E}(s)\}$ of (3.23) satisfies $\hat{E}(s) \in T_{\sigma(s)}(A(p_0))$ for all $s \in [0, \ell]$.

Observe that the parallel vector fields are uniquely determined by their initial values. Hence, if $E(0) = \hat{E}(0) \in T_{\sigma(0)}(A(p_0))$, then $E(s) = \hat{E}(s) \in T_{\sigma(s)}(A(p_0))$ for all $s \in [0, \ell]$. This completes the proof of Assertion 3.13 as well as Theorem E. $\square$

Finally, we should point out that the conclusion of Theorem E fails if the assumption of nonnegative sectional curvature is dropped.

**Example 3.14** (Compare [BG] Section 4). Let $\tilde{M}^2 = \{(x, y) | x^2 + y^2 < 1\}$ with the Poicaré metric and $\tilde{\mathcal{S}} = \{(x, 0) | |x| < 1\}$. For each $x \in \tilde{\mathcal{S}}$, we define $F_x = F_x^+ \cup F_x^-$ where $F_x^+ = \{(t, y) | [t - \frac{1}{2}(x+1)]^2 + y^2 = \frac{1}{4}(1-x)^2, y \geq 0\}$ is the upper half horo-circle with the center $(\frac{1}{2} + \frac{x}{2}, 0)$ and $F_x^- = \{(t, y) | [t - \frac{1}{2}(x-1)]^2 + y^2 = \frac{1}{4}(1+x)^2, y \leq 0\}$

is the lower half horo-circle with the center $(-\frac{1}{2}+\frac{x}{2},0)$. $F_x^+$ meets $F_x^-$ at $(x,0)$ $C^{1,1}$-smoothly, but not $C^2$-smoothly.

With respect to Poicaré metric, the curves $\{F_x\}$ form an 1-family of equidistant curves. Thus, there exists a corresponding Riemannian submetric map $\tilde{\Psi}:\tilde{M}^2\to\tilde{\mathcal{S}}$ such that $\tilde{\Psi}^{-1}\big((x,0)\big)=F_x$ for all $x$.

Let $0<a<1$, there is an isometry of $\tilde{M}^2$ given by

$$f_a(x+\sqrt{-1}y)=\frac{x-a+\sqrt{-1}y}{1-a(x+\sqrt{-1}y)}$$

The isometry $f_a$ preserves the geodesic $\tilde{\mathcal{S}}$ and keeps the family $\{F_x\}$ invariant. Let $\mathbb{Z}$ be the group generated by $f_a$. Consider the quotient spaces $M^2=\tilde{M}^2/\mathbb{Z}$ and $\mathcal{S}=\tilde{\mathcal{S}}/\mathbb{Z}$. Then $\mathcal{S}$ is the unique closed geodesic within its free homotopy class in $M^2$. This induces an Riemannian submetric map $\Psi:M^2\to\mathcal{S}$.

Notice that $F_x^+$ meets $F_x^-$ at $(x,0)$ only $C^{1,1}$-smoothly. Hence, $\Psi$ is exactly $C^{1,1}$-smooth, but not $C^2$-smooth.

Similarly, one can consider one sheet hyperboloid $M^2$ given by

$$M^2=\{(x,y,z)|x^2+y^2=1+z^2\}.$$

The set $\mathcal{S}=\{(x,y,0)|x^2+y^2=1\}$ is unique non-trivial closed geodesic in $M^2$. Clearly, $M^2$ has non-positive curvature. Using the same method as above, one can construct a $C^{1,1}$-smooth Riemannian submetric map $\Psi:M^2\to\mathcal{S}$ such that $\Psi$ is not $C^2$-smooth.

We should point out that, in the examples above, there is no compatible Fermi maps associated with $\Psi$. Therefore, as Example 3.14 indicated, the assumption of nonnegative section curvature is essential to construct compatible vertical Fermi maps in our proof of Theorem E.

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