THE SMOOTHNESS OF RIEMANNIAN SUBMERSIONS WITH NONNEGATIVE SECTIONAL CURVATURE

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In this article, we study the smoothness of Riemannian submersions for open manifolds with non-negative sectional curvature. Suppose that M^n is a C^∞-smooth, complete and non-compact Riemannian manifold with nonnegative sectional curvature. Cheeger-Gromoll [ChG] established a fundamental theory for such a manifold. Among other things, they showed that M^n admits a totally convex exhaustion $\{\Omega_u\}_{u\geq 0}$ of M^n, where $\Omega_0 = \mathcal{S}$ is a totally geodesic and compact submanifold without boundary. Furthermore, M^n is diffeomorphic to the normal vector bundle of the soul $\mathcal{S}$.

Sharafutdinov found that there exists a distance non-increasing retraction $\Psi : M^n \to \mathcal{S}$ from the open manifold M^n of non-negative sectional curvature to its soul, (cf. [Sh], [Y2]). Perelman [Per] further showed that such a map Ψ is indeed a C^1-smooth Riemannian submersion. Furthermore, $\Psi[Exp_q(t\vec{v})] = q$ for any $q \in \mathcal{S}$ and $\vec{v} \perp T_q(\mathcal{S})$. Therefore, the fiber $F_q = \Psi^{-1}(q)$ is a k-dimensional submanifold, which is C^∞-smooth almost everywhere, where $k = dim(M^n) - dim(\mathcal{S}) > 0$.

Guijarro [Gu] proved that the fiber F_q is indeed a C^2-smooth submanifold for each $q \in S$. In this paper, we prove that the fibres are C^∞-smooth.

Theorem 1. *Let M^n be a complete, non-compact and C^∞-smooth Riemannian manifold with nonnegative sectional curvature. Suppose $\mathcal{S}$ is a soul of M^n. Then any distance non-increasing retraction $\Psi : M^n \to \mathcal{S}$ must give rise to a C^∞-smooth Riemannian submersion with the property that*

$$\Psi[Exp_q(\mathbb{R}\{\vec{v}\})] = q$$

for all $q \in \mathcal{S}$ and $\vec{v} \perp \mathcal{S}$, where Exp_q is the exponential map of M^n at q.

Professor Wilking kindly informed us that he has recently obtained the similar result (cf. [Wi]). His method is completely independent of ours. Our proof of

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Theorem 1 uses a flat strip theorem associated with Cheeger-Gromoll exhaustion (cf. [CaS] or Theorem 4 below), an uniform estimate for cut-radii of convex subsets in [ChG] and a bootstrap argument using the smoothness of geodesic flow.

For each compact convex subset $\Omega \subset M^n$, we let $U_\epsilon(\Omega) = \{x \in M^n | d(x, \Omega) < \epsilon\}$. Its cut-radius is given by $\delta_\Omega = \sup\{\epsilon |$ there is a unique nearest point projection $\mathcal{P}_\Omega : U_\epsilon(\Omega) \to \Omega\}$.

For each $x \in M^n$, we let $\mathrm{Inj}_{M^n}(x)$ be the injectivity radius of M^n at x. Similarly, let $\mathrm{Inj}_{M^n}(A) = \{\mathrm{Inj}_{M^n}(x) | x \in A\}$. We need the following semi-global estimate for cut-radius. Recall that there is a totally convex exhaustion $\{\Omega_u\}_{u \geq 0}$ of M^n given in [ChG].

Lemma 2. *(Lemma 2.4 of [ChG], [CaS]) Let $A \subset \Omega_T$ be a connected, convex and compact subset in a Riemannian manifold M^n with nonnegative curvature, let $K_0 = \max\{K(x) | x \in \Omega_{T+1}\}$, $Inj_{M^n}(\Omega_T)$ be the upper bound of sectional curvature on Ω_{T+1} and $\mathcal{S}$ be as above. Suppose that $\dim(\Omega_T) = n$. Then the subset A has cut-radius bounded below by*

$$\delta_A \geq \delta_0(T) = \frac{1}{4} \min\{Inj_{M^n}(\Omega_T), \frac{\pi}{\sqrt{K_0}}, 1\},$$

where $\delta_0(T)$ is independent of choices of A with $A \subset \Omega_T$.

Let us briefly recall the Cheeger-Gromoll convex exhaustion. According to [ChG], there is a partition $a_0 = 0 < a_1 < ...a_m < a_{m+1} = \infty$ of $[0, \infty)$ and an exhaustion of $\{\Omega_u\}_{u \geq 0}$ of M^n such that the following holds:

(1) $M^n = \cup_{u \geq 0} \Omega_u$. If $u > a_m$ then $dim[\Omega_u] = n$. If $u \leq a_m$, then $\dim[\Omega_u] < n$.

(2) $\Omega_0 = \mathcal{S}$ is the soul of M^n, which is a totally geodesic C^∞-smooth compact submanifold without boundary.

(3) If $u > 0$, Ω_u is a totally convex, compact subset of M^n and hence Ω_u is a compact submanifold with a C^∞-smooth relative interior. Furthermore, $\dim(\Omega_u) = k_u > 0$ and Ω_u has a non-empty $(k_u - 1)$-dimensional relative boundary $\partial \Omega_u$;

(4) For any $u_0 \in [a_j, a_{j+1}]$ and $0 \leq t \leq u_0 - a_j$, the family $\{\Omega_{u_0 - t}\}_{t \in [0, u_0 - a_j]}$ is given by *the inward equidistant evolution*:

$$\Omega_{u_0 - t} = \{x \in \Omega_{u_0} | d(x, \partial \Omega_{u_0}) \geq t\}. \tag{2.1}$$

(5) If $u > a_m$ then $u - a_m = \max\{d(x, \partial \Omega_u) | x \in \Omega_u)\}$. If $0 \leq j \leq m - 1$ then $a_{j+1} - a_j = \max\{d(x, \partial \Omega_{a_{j+1}}) | x \in \partial \Omega_{a_{j+1}}\}$ and hence $\dim[\Omega_{a_j}] < \dim[\Omega_{a_{j+1}}]$ for $j \geq 0$.

Assume that $k = \dim[M^n] - \dim[\mathcal{S}] = \dim(F_q)$ for all $q \in \mathcal{S}$. Since $M^n = \cup_{T \geq 0} \Omega_T$, it is sufficient to verify that the subset $[U_{\delta_0(T)}(\Omega_T) \cap F_q]$ has a k-dimensional C^∞-smooth interior, where $\delta_0(T)$ is given by Lemma 2 and $T > a_m$.

For this purpose, we need to study the geometry of the equidistant hypersurfaces from $\partial\Omega_u$. Federer [Fe] has studied the smoothness of *the outward equidistant hypersurfaces* $\partial[U_\epsilon(\Omega)]$ for $0 < \epsilon < \delta_\Omega$. Following his approach, we consider the outward normal cone of Ω as follows:

$$\mathcal{N}^+(\Omega, M^m) = \{(p,\vec{v}) | p \in \Omega, d(Exp_p(t\vec{v}), \Omega) = t|\vec{v}|, \text{ for } 0 \le t|\vec{v}| < \delta_\Omega\}.$$

If $\{\Omega_u\}$ is the Cheeger-Gromoll inward exhaustion as above and $u > 0$, then the relative boundary $\partial\Omega_u$ is not necessarily smooth. We are going to study the corresponding decomposition of $\mathcal{N}^+(\Omega, M^m)$:

$$\mathcal{N}_p^+(\Omega_u, M^n) \subset \big[\mathcal{N}_p^+(\Omega_u, \text{int}(\Omega_{u+\epsilon})) \oplus \mathcal{N}_p^+(\text{int}(\Omega_{u+\epsilon}), M^n)\big], \tag{2.2}$$

where $\mathcal{N}_p^+(\Omega_u, \text{int}(\Omega_{u+\epsilon}))$ is defined by

$$\begin{aligned}\mathcal{N}^+\big(\Omega_u, \text{int}(\Omega_{u+\epsilon})\big) =&\{(p,\vec{v}) | p \in \Omega_u, d(Exp_p(t\vec{v}), \Omega_u) = t|\vec{v}|,\\ &\text{for } 0 \le t|\vec{v}| < \delta_{\Omega_u}, Exp_p(t\vec{v}) \in \text{int}(\Omega_{u+\epsilon})\}.\end{aligned}$$

Our next step is to choose ϵ sufficiently small so that (1) there is a nearest point projection $\mathcal{P} : \text{int}(\Omega_{u+\epsilon}) \to \Omega_u$; and (2) $\Omega_u = \{x \in \Omega_{u+\epsilon} | d(x, \partial\Omega_{u+\epsilon}) \ge \epsilon\}$ holds. We first find j so that $a_j \le u < a_{j+1}$ for some $0 \le j \le m$. Let $T = u + a_m + 1$ and $\delta_0(T)$ be given by Lemma 2. It follows from a result of Yim that there is a constant C_T such that, for $0 \le u_1 < u_2 \le T$, we have

$$\max\{d(x, \Omega_{u_1}) | x \in \Omega_{u_2}\} \le C_T(u_2 - u_1), \tag{2.3}$$

see [Y2, Theorem A.5(3)].

In what follows, we always choose

$$0 < \epsilon = \epsilon_u < \min\{[a_{j+1} - u], \frac{\delta_0(T)}{2C_T}\}, \tag{2.4}$$

where $u \in [a_j, a_{j+1})$, $T = u + a_m + 1$ and $\delta_0(T)$ is given by Lemma 2.

With such a choice of $\epsilon = \epsilon_u$ by (2.4), the geometry of $\mathcal{N}_p^+(\Omega_u, \text{int}(\Omega_{u+\epsilon}))$ is determined by its minimal normal vectors which we now describe.

Definition 3. (Minimal normal vector) Let Ω_u, $\Omega_{u+\epsilon}$ and $\mathcal{N}^+(\Omega_u, \text{int}(\Omega_{u+\epsilon}))$ be as above. Let $\sigma_{(p,\vec{v})} : [0,\epsilon] \to M^n$ be a geodesic given by $\sigma_{(p,\vec{v})}(t) = \text{Exp}_p(t\frac{\vec{v}}{|\vec{v}|})$, where $\vec{v} \ne 0$. If $\sigma_{(p,\vec{v})}$ is a length-minimizing geodesic from $p \in \Omega_u$ to $\partial\Omega_{u+\epsilon}$, then $\vec{v}$ is called a minimal normal vector in $\mathcal{N}_p^+(\Omega_u, \text{int}(\Omega_{u+\epsilon}))$.

It is known that any other normal vector $\vec{w} \in \mathcal{N}_p^+(\Omega_u, \text{int}(\Omega_{u+\epsilon}))$ can be expressed as a linear combination of *minimal normal vectors* at p. Moreover, the convex hull of minimal normal vectors at p is equal to $\mathcal{N}_p^+(\Omega_u, \text{int}(\Omega_{u+\epsilon}))$, (cf. [Y1, Proposition 1.7]).

For each $p \in M^n$, we let $\mathcal{V}_p = T_p(F_{\Psi(p)})$ and $\mathcal{H}_p = [\mathcal{V}_p]^\perp$. A geodesic $\alpha : [a,b] \to M^n$ is said to be horizontal, if $\alpha'(t) \perp F_{\Psi(\alpha(t))}$ for all $t \in [a,b]$. We need the following flat strip theorem for the proof of Theorem 1.

Theorem 4. *Let $\{\Omega_u\}$ be the Cheeger-Gromoll totally convex exhaustion of M^n as above. Suppose that $\Psi : M^n \to \mathcal{S}$ be a distance non-increasing retraction and $F_q = \Psi^{-1}(q)$ be a fibre for some $q \in \mathcal{S}$. Then for $p \in F_q \cap \Omega_u$ and any $(p, \vec{v}) \in \mathcal{N}^+(\Omega_u, M^n)$, we have*

$$\Psi[Exp_p([\mathbb{R}\{\vec{v}\}])] = \Psi(p) = q. \tag{4.1}$$

Moreover, if $\dim(\mathcal{S}) \geq 1$ and if $\vec{w} \in \mathcal{H}_p$ has $|\vec{w}| = 1 = |\vec{v}|$, then the surface $\Sigma^2_{\vec{v},\vec{w}} = Exp_p[\mathbb{R}\{\vec{v}\} \oplus \mathbb{R}\{\vec{w}\}]$ is totally geodesic immersed flat plane in M^n.

A result similar to Theorem 4 was proved in [CaS] via a totally different approach, which involves the barrier surface technique inspired by the work of Calabi.

Proof of Theorem 4. Theorem 4 was proved by Perelman [Per] for the case of $\Omega_0 = \mathcal{S}$. Applying Perelman's argument for the case of $p \notin \mathcal{S}$, Guijarro [Gu1] found the following sufficient condition for (4.1).

(4.2) *$\vec{v} \in \mathcal{V}_p$ stays vertical under parallel transport along any horizontal broken geodesic.*

Guijarro showed that (4.1) follows from (4.2). Moreover, if (4.2) holds and if $\vec{w} \in \mathcal{H}_p$ has $|\vec{w}| = 1 = |\vec{v}|$, then the surface $\Sigma^2_{\vec{v},\vec{w}} = Exp_p[\mathbb{R}\{\vec{v}\} \oplus \mathbb{R}\{\vec{w}\}]$ is totally geodesic immersed flat plane in M^n, (cf. Theorem 3.1 of [Gu1]).

In order to see that $\mathcal{N}^+(\Omega_u, M^n) \subset \mathcal{V}_p$ holds, we also need

(4.3) ([Gu2, Corollary 3.2]) *Any horizontal geodesic α with $\alpha(0) \in \partial\Omega_u$ must be entirely contained in $\partial\Omega_u$. Consequently, $\mathcal{H}_p \subset T^-_p(\partial\Omega_u)$, where $T^-_p(\partial\Omega_u)$ is the tangent cone of $\partial\Omega_u$ at p*

For the convenience of the reader, we include Guijarro's short proof here. Let $\ell = d(\alpha(0), \mathcal{S})$. It follows from Perelman's Theorem that $\alpha(\mathbb{R}) \subset \bar{U}_\ell(\mathcal{S})$. Choose sufficient large $N > a_m$ so that $\Omega_N \supset \bar{U}_{\ell+1}(\mathcal{S})$. By Theorem 1.10 of [ChG], the distance function $r_N(y) = d(y, \partial\Omega_N)$ is convex for $y \in \Omega_N$. The function $h_N(s) = r_N(\alpha(s))$ is a bounded convex function for $s \in \mathbb{R}$ and hence h must be a constant function. For $u > a_m$, then $h_N(s) \equiv c = [N - u]$ and hence $\alpha(\mathbb{R}) \subset \partial\Omega_{N-(N-u)} = \partial\Omega_u$. For $u \leq a_m$, we have $h_N(s) \equiv c = [N - a_m]$ and $\alpha(\mathbb{R}) \subset \Omega_{a_m}$. Let j be the integer such that $a_{j+1} \geq u > a_j$. Consider the distance functions $r_{a_i}(y) = d(y, \partial\Omega_{a_i})$ for $i = m, m-1, ..., j+1$. For the same reason as above, one can show that the assertion (4.3) holds for any u.

Let $T = u + a_m + 1$ and $\delta_0(T)$ and ϵ be in (2.4) as above. Recall that by (2.2) we have

$$\mathcal{N}^+_p(\Omega_u, M^n) \subset \big[\mathcal{N}^+_p(\Omega_u, \mathrm{int}(\Omega_{u+\epsilon})) \oplus \mathcal{N}^+_p(\mathrm{int}(\Omega_{u+\epsilon}), M^n)\big].$$

For $\vec{v}$ in either $\mathcal{N}^+_p(\Omega_u, \mathrm{int}(\Omega_{u+\epsilon}))$ or $\mathcal{N}^+_p(\mathrm{int}(\Omega_{u+\epsilon}), M^n)$, we will show that such a $\vec{v}$ satisfies (4.2).

It follows from Theorem 1.10 of [ChG] (or Corollary 1.4 of [Y1]) that any *minimal normal vector* $\vec{v}$ of $\mathcal{N}_p^+(\Omega_u, \mathrm{int}(\Omega_{u+\epsilon}))$ stays *minimal* under parallel transport along any geodesic in $\partial\Omega_u$. Hence, the bundle $\mathcal{N}^+(\Omega_u, \mathrm{int}(\Omega_{u+\epsilon}))$ is invariant under parallel transport along any geodesic in $\partial\Omega_u$. This together with (4.3) implies that if $\vec{v} \in \mathcal{N}_p^+\big(\Omega_u, \mathrm{int}(\Omega_{u+\epsilon})\big)$ then $\vec{v}$ satisfies (4.2).

For $\vec{v} \in \mathcal{N}_p^+(\mathrm{int}(\Omega_{u+\epsilon}), M^n)$, the assertion (4.2) follows from Corollary 3.2 of [Gu1]. In fact, since $\mathrm{int}(\Omega_{u+\epsilon})$ is totally convex and totally geodesic, both $T(\mathrm{int}(\Omega_{u+\epsilon}))$ and $N^+(\mathrm{int}(\Omega_{u+\epsilon}), M^n)$ are invariant under parallel transport along any geodesic in $\mathrm{int}(\Omega_{u+\epsilon})$. This together with (4.3) implies that (4.2) holds for any $\vec{v} \in \mathcal{N}_p^+(\mathrm{int}(\Omega_{u+\epsilon}), M^n)$.

Therefore, (4.2) holds for any $\vec{v} \in \mathcal{N}_p^+(\Omega_u, M^n)$. This completes the proof of Theorem 4. $\square$

For the proof of Theorem 1, we consider the geodesic flow of the unit tangent sphere bundle

$$SM^n = \{(p, \vec{v}) | p \in M^n, \vec{v} \in T_p(M^n), |\vec{v}| = 1\}.$$

The geodesic flow $\tau_t : SM^n \to SM^n$ is given by $\tau_t(p, \vec{v}) = \big(Exp_p(t\vec{v}), \frac{d[Exp_p(t\vec{v})]}{dt}\big)$. When the metric of M^n is C^∞-smooth, τ_t is a C^∞-smooth diffeomorphism.

With Lemma 2, Theorem 4 and the smoothness of the geodesic flow, we are ready to prove Theorem 1.

Proof of Theorem 1. As we pointed above, it is sufficient to verify that the subset $[U_{\delta_0(T)}(\Omega_T) \cap F_q]$ has a k-dimensional C^∞-smooth interior for any given $T > a_m$ and $q \in \mathcal{S}$, where $\delta_0(T)$ is given by Lemma 2.

Fix $T > a_m$ with $\dim[\Omega_T] = n$. For each $u \in [0, T]$, there is a distance function $r_u(y) = d(y, \Omega_u)$. Let $\mathrm{int}(\Omega_u) = [\bar{\Omega}_u - \partial\Omega_u]$ be the relative interior of Ω_u. Recall that $\mathrm{int}(\Omega_u)$ is a totally convex and totally geodesic submanifold of M^n. Hence, $\mathrm{int}(\Omega_u)$ is a C^∞-smooth submanifold of M^n. Although the level set $r_u^{-1}(\{\delta\})$ may not be C^∞-smooth, the sub-level set $r_u^{-1}\big((-\infty, \delta)\big)$ is always an open C^∞-smooth domain of M^n. We are going to verify the following assertion:

Assertion (5.u). *The sub-level set $[U_{\delta_0(T)}(\Omega_u) \cap F_q]$ has the k-dimensional C^∞-smooth interior, where $q \in \mathcal{S}$ and $k = \dim[F_q]$.*

Since the soul $\mathcal{S}$ has the cut radius $\geq \delta_0(T)$ and $\mathcal{S}$ is C^∞-smooth, the assertion (5.u) holds for $u = 0$ and $\Omega_0 = \mathcal{S}$ by Perelman's theorem.

Our next step is to show that the assertion (5.u) holds for $0 \leq u \leq \frac{\delta_0(T)}{2C_T}$, where C_T is given by (2.3).

Since Ω_u is a compact subset and $U_{\delta_0(T)}(\mathcal{S})$ is an open subset in M^n, we see that $\ell_u = \max\{d(x, \mathcal{S}) | x \in \Omega_u\} \leq \frac{\delta_0(T)}{2}$. Hence, we have $\epsilon_0 = \delta_0(T) - \ell_u > 0$.

For each $x \in [U_{\delta_0(T)}(\Omega_u) \cap F_q - U_{\delta_0(T)}(\mathcal{S})]$, we are going to choose a k-dimensional open neighborhood $V_\epsilon(x)$ of x in F_q and a C^∞-smooth coordinate system for $V_\epsilon(x)$.

The construction of $V_\epsilon(x)$ takes two steps. Let $\mathcal{P}_u : U_{\delta_0(T)}(\Omega_u) \to \Omega_u$ be the nearest point projection from M^n to Ω_u and $y_x = \mathcal{P}_u(x)$. Suppose that $\sigma : [0, \ell_x] \to$

M^n is the unique shortest geodesic segment of unit speed from y_x to x with $\sigma(0) = y_x$ and $\sigma(\ell_x) = x$. Since $\sigma'(0) \in \mathcal{N}^+(\Omega_u, M^n)$, it follows from Theorem 4 that σ is entirely contained in F_q.

Let us now begin to construct $V_\epsilon(x)$. Recall that $\epsilon_0 = \delta_0(T) - \ell_u > 0$. We choose $0 < \epsilon_1 < \epsilon_2 < \epsilon_0$ and an annular region:

$$A(\Omega_u, \epsilon_1, \epsilon_2) = \{z \in M^n | \epsilon_1 < d(z, \Omega_u) < \epsilon_2\}$$

It is clear that $A(\Omega_u, \epsilon_1, \epsilon_2) \subset U_{\delta_0(T)}(\mathcal{S})$. We already showed that $F_q \cap A(\Omega_u, \epsilon_1, \epsilon_2) \subset F_q \cap U_{\delta_0(T)}(\mathcal{S})$ is C^∞-smooth k-dimensional open sub-manifold.

Let us choose ϵ_3 with $\epsilon_1 < \epsilon_3 < \epsilon_2$ and $\hat{x} = \sigma(\epsilon_3)$. We consider $r(y) = d(y, \Omega_u)$. By Corollary 5.3 of [Fed], the level set $[r^{-1}(\epsilon_3) \cap F_q]$ is a $C^{1,1}$-smooth $(k-1)$-dimensional hypersurface in $[F_q \cap A(\Omega_u, \epsilon_1, \epsilon_2)]$. The integral curves of the gradient of ∇r are portions of normal geodesics from Ω_u. It follows from Theorem 4 that those integral curves (with initial points in $[F_q \cap A(\Omega_u, \epsilon_1, \epsilon_2)]$) must be entirely contained in F_q. Hence, ∇r is tangent to F_q on the smooth subset $[F_q \cap A(\Omega_u, \epsilon_1, \epsilon_2)]$. Let $\Sigma^k = [F_q \cap A(\Omega_u, \epsilon_1, \epsilon_2)]$ and $B^{k-1}(0)$ be the unit ball in $\mathbb{R}^{k-1}$. It is well-known (cf. [Hi]) that the mapping space $C^\infty(B^{k-1}(0), \Sigma^k)$ is dense in another mapping space $C^{1,1}(B^{k-1}(0), \Sigma^k)$. Therefore, there exist an element $h \in C^\infty(B^{k-1}(0), \Sigma^k)$ and a C^∞-smooth hypersurface $W^{k-1} = h(B^{k-1}(0))$ such that (1) the smooth hypersurface W^{k-1} is C^1-close to the $C^{1,1}$-hypersurface $[r^{-1}(\epsilon_3) \cap F_q \cap U_{\epsilon_4}(\hat{x})]$ in a sufficiently small neighborhood $F_q \cap U_{\epsilon_4}(\hat{x})$ of $\hat{x}$; and (2) W^{k-1} is transversal to the vector field ∇r everywhere. Thus, there is a C^∞-smooth local coordinate system around $\hat{x}$:

$$\hat{h} : B_\epsilon^{k-1}(0) \times (t_1, t_2) \to \hat{V}_\epsilon(\hat{x})$$

such that $\frac{\partial \hat{h}}{\partial t} = \nabla r$ and the curve $t \to \hat{h}((w_1, ..., w_{k-1}), t)$ is a integral curve of ∇r. The vector $\frac{\partial \hat{h}}{\partial w_j}$ is transversal to $\frac{\partial \hat{h}}{\partial t}$, but $\frac{\partial \hat{h}}{\partial w_j}$ is not necessarily orthogonal to $\frac{\partial \hat{h}}{\partial t}$ for $j = 1, 2, ..., k-1$.

Notice that $\hat{V}' = \{(\hat{h}(y), \frac{\partial \hat{h}}{\partial t}(y)) | y \in B_\epsilon^{k-1}(0) \times (t_1, t_2)\}$ is a k-dimensional, C^∞-smooth submanifold of SM^n. If $c = d(x, \hat{x})$, then $\tau_c(\hat{V}')$ remains to be a k-dimensional C^∞-smooth submanifold of SM^n, where $\{\tau_t\}_{t \in \mathbb{R}}$ is the geodesic flow of the unit tangent sphere bundle SM^n. We may assume that $\epsilon_3 + c + |t_2 - t_1| < \delta_0(T)$ by choosing $|t_2 - t_1|$ sufficiently small.

Let $\pi : SM^n \to M^n$ be the foot-point projection given by $\pi(p, \vec{v}) = p$. By Theorem 4, we have $\tau_c(\hat{V}') \subset SF_q$, where SF_q is the unit tangent sphere bundle of the fibre F_q. Since $\epsilon_3 + c + |t_2 - t_1| < \delta_0(T)$ and Ω_u has cut-radius $\geq \delta_0(T)$, the restriction $\pi|_{\tau_c(\hat{V}')} : \tau_c(\hat{V}') \to F_q$ is injective.

Setting $V_\epsilon(x) = \pi[\tau_c(\hat{V}')]$, we are going to show that $V_\epsilon(x)$ is diffeomorphic to $\tau_c(\hat{V}')$ via the map $\pi|_{\tau_c(\hat{V}')}$. Let $d\pi$ be the derivative of the map π. It is sufficient to $d\pi(\vec{\xi}) \neq 0$ for any non-zero vector $\vec{\xi}$ tangent to $\tau_c(\hat{V}')$. It is well-known that the variation vector field of geodesics are Jacobi fields along a geodesic.

Thus, the tangent vectors of $\tau_c(\hat{V}')$ at an element $(\tilde{\sigma}(t), \tilde{\sigma}'(t)) \in \tau_c(\hat{V}')$ are in the form of $\vec{\xi} = (J(t), J'(t))$ for some Jacobi field J along a geodesic $\tilde{\sigma}$. Hence, $d\pi((J(t), J'(t)) = J(t)$. By our construction, $\tau_c(\hat{V}')$ is foliated by the geodesics $\{\tilde{\sigma}\}$ with $(\tilde{\sigma}(0), \tilde{\sigma}'(0)) \in \mathcal{N}^+(\Omega_u, M^n)$. Since Ω_u has no focal points in $V_\epsilon(x)$, for $(\tilde{\sigma}(t), \tilde{\sigma}'(t)) \in \tau_c(\hat{V}')$, we have $d\pi(J(t), J'(t)) = J(t) \neq 0$ for any non-trivial tangent vector $\vec{\xi} = (J(t), J'(t))$ of $\tau_c(\hat{V}')$ at $(\tilde{\sigma}(t), \tilde{\sigma}'(t))$. Thus, we conclude that $\pi|_{\tau_c(\hat{V}')}$ is a C^∞-smooth diffeomorphism between $\tau_c(\hat{V}')$ and $V_\epsilon(x)$. Therefore, x has a C^∞-smooth open neighborhood $V_\epsilon(x)$ in F_q. In summary, we have proved the following fact

Fact 6. (Step 1 in the bootstrap argument) *If the subset $[U_{\delta_0(T)}(\mathcal{S}) \cap F_q]$ has the C^∞-smooth, k-dimensional interior and if $\Omega_u \subset U_{\delta_0(T)}(\mathcal{S})$, then the corresponding tubular neighborhood $[U_{\delta_0(T)}(\Omega_u) \cap F_q]$ of Ω_u (with the same radius) has the C^∞-smooth, k-dimensional interior as well. In particular, (5.u) holds for $u = u_1$ with $0 \leq u_1 \leq \frac{\delta_0(T)}{2C_T}$.*

The proof of Fact 6 can be elaborated to verify the following.

Fact 7. (The j-th step in the bootstrap argument) *If the subset $[U_{\delta_0(T)}(\Omega_{u_j}) \cap F_q]$ has C^∞-smooth, k-dimensional interior and if $\Omega_u \subset U_{\delta_0(T)}(\Omega_{u_j})$, then the corresponding tubular neighborhood $[U_{\delta_0(T)}(\Omega_u) \cap F_q]$ of Ω_u (with the same radius) has C^∞-smooth, k-dimensional interior as well. In particular, (5.u) holds for $u = u_{j+1}$ with $u_j \leq u_{j+1} \leq u_j + \frac{\delta_0(T)}{2C_T}$.*

Choose $N \geq 2C_T[\frac{T}{\delta_0(T)} + 1]$. Applying the above Fact 7 repeatedly (at most N times), one can complete the proof of (5.u) for $u = T$. Since $M^n = \cup_T \Omega_T$, Theorem 1 follows from (5.T) for any arbitrarily large T. $\square$

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