VERONESE SURFACES AND LINE ARRANGEMENTS

TOM GRABER AND KRISTIAN RANESTAD*

Abstract. We formulate a generalization of Castelnuovo‘s lemma on rational normal curves to d-uple Veronese surfaces and apply it to define a Hilbert scheme compactification of line arrangements in the Hilbert scheme of Veronese surfaces. In the case of arrangements of five lines, we find the compactification to be isomorphic to a Del Pezzo surface of degree 5. We thus recover previous results of Kapranov and Hacking.

Paul Hacking recently studied compact moduli of hyperplane arrangements using stable pairs of a possibly reducible variety and a divisor on it [1]. His compactification coincides with a Hilbert scheme compactification of so called special Veronese varieties due to Kapranov [2].

In this note we consider line arrangements in the plane and define what coincides with Kapranov's special Veronese varieties of dimension 2. In particular, we recover an analogue of Castelnuovo's lemma on rational normal curves for $d-$uple Veronese surfaces. This sets up a natural map of Hacking's moduli space of line arrangements of degree $d+3$ to the Hilbert scheme of $d-$uple Veronese surfaces. We give a different and direct proof that the corresponding Hilbert scheme compactification of the moduli of stable 5-line arrangements is naturally isomorphic to a smooth Del Pezzo surface of degree 5. Our enumeration of singular surfaces in the family, recovers Hacking's classification.

1. A Castelnuovo type lemma

We start with an analogue of

Lemma 1.1. *[Castelnuovo] Through $d+3$ points in linear general position in $\mathbb{P}^d$ there is a unique rational normal curve.*

due to Kapranov:

Lemma 1.2. [2] *Given $d+4$ general $\mathbb{P}^d$'s in P^N with $N=(d+2)(d+1)/2-1$, such that any two of the $\mathbb{P}^d$'s meet in exactly a point, and any set of $d+1$ intersection points span a $\mathbb{P}^d$. Then there is a unique d-uple Veronese surface that contains the $(d+4)(d+3)/2$ intersection points.*

Remark 1.3. *Note that we need the first "general", since our proof is not constructive.*

Proof. First we assume that V is a $d-$uple Veronese surface through all the intersection points. By Castelnuovo's lemma, in each $\mathbb{P}^d$ there is a unique rational normal curve of degree d through the $d+3$ intersection points. We show that V actually

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must intersect each $\mathbb{P}^d$ in the unique rational normal curve given by Castelnuovo's lemma. For this, fix a $\mathbb{P}^d$ and let Γ be the $d+3$ intersection points in it. Let C be a general curve in the linear system $|(d+3)L|$ on V that contains Γ, where L is the class of a line in $V \cong \mathbb{P}^2$. Since Γ is smooth, C is smooth. In fact $d+3$ lines through the $d+3$ points generate a linear system without base points outside Γ, so by Bertini C is smooth. Since $C \subset V$ is the canonical embedding, Γ moves in a net on C. But C has only one g^2_{d+3}, so Γ must therefore coincide with the $L \cap C$ for some rational normal curve $L_0 \in |L|$ of degree d on V. Clearly L_0 coincides with the unique rational normal curve given by Castelnuovo's lemma. Thus V contains $d+4$ rational normal curves of degree d.

On the other hand any set of $d+4$ general members of the linear system $|L|$ on V gives rise to $d+4$ pairwise intersecting $\mathbb{P}^d$'s in $\mathbb{P}^N$.

Now, consider the family of marked Veronese surfaces, where the marking is a line arrangement of degree $d+4$, i.e. a collection of $d+4$ lines in the plane. Clearly this family is irreducible. We compute the dimension by adding the dimension of the orbit of Veronese surfaces to the dimension of the family of line arrangements and get

$$(d+2)^2(d+1)^2/4 - 9 + 2(d+4) = d(d+4)(d^2+2d+5)/4$$

Compare this with the family of sets of $d+4$ pairwise intersecting $\mathbb{P}^d$'s. This family is clearly also irreducible. We compute its dimension as a configuration of $d = 4$ points in the Grassmannian satisfying the relevant Schubert condition and get

$$(d+4)(d+1)((d+2)(d+1)/2-(d+1))-((d+2)(d+1)/2-2(d+1)+1)(1+2+...+d+3)$$
$$= d(d+4)((d+1)^2/2 - (d-1)(d+3)/4) = d(d+4)(d^2+2d+5)/4.$$

Since the two families have the same dimension, the lemma follows as soon as the Veronese surface is unique for a general configuration of d-planes. For this, let C be the line arrangement of degree $d+4$. The d-th Veronese embedding is given by $K_C - L_C$, where L_C is the divisor of the embedding in $\mathbb{P}^2$. The divisor L_C is unique (given by the divisors determined by its d+4 line components), so $K_C - L_C$ is unique, and therefore also the Veronese surface. □

2. Compactifying the muduli of stable line arrangements

Consider the family Y_d of $d-$uple Veronese surfaces that intersect $d+3$ pairwise intersection $\mathbb{P}^d$'s in a rational normal curve. Now, the family X_d of $\mathbb{P}^d$'s that intersect each of the $d+3$ fixed $\mathbb{P}^d$'s has dimension $2d$ in the Grassmannian.

Note that, by Lemma 1.2, there is a rational map $X_d \to Y_d$ whose general fiber is a (dual) Veronese surface. Thus

Proposition 2.1. *The family Y_d of $d-$uple Veronese surfaces that intersect $d+3$ pairwise intersecting $\mathbb{P}^d$'s in a rational normal curve, has dimension $2d-2$.*

Definition 2.2. *A line arrangement of degree $d > 3$ is called stable, if no three lines are concurrent.*

The family of stable line arrangements of degree $d+3$ modulo automorphisms has dimension $2d-2$. Now, clearly any Veronese surface in Y_d defines a stable line arrangement of degree $d+3$.

Lemma 2.3. *Two Veronese surfaces in Y_d determine distinct stable line arrangements of degree $d+3$.*

Proof. The embedding of a line arrangement of degree $d+3$ on a d-uple Veronese surface is given by the canonical linear series, so the Veronese surface is unique. □

Therefore the moduli of stable line arrangements has a compactification in the Hilbert scheme of d-uple Veronese surfaces. More precisely, we define Y_d^H to be the reduced closure of Y_d in the Hilbert scheme of Veronese surfaces.

Here we treat the case of $d = 2$. So we consider the family of Veronese surfaces in $\mathbb{P}^5$, defined according to the lemma by fixing five $\mathbb{P}^2$'s and letting a sixth plane P move in the family of planes that meet all the first five. This family of planes form a subvariety X in $\mathbf{Gr}(3,6)$ of dimension 4, since it is the proper intersection of five Schubert hyperplane sections, while the corresponding family Y_2 of Veronese surfaces has dimension 2.

Let us show directly how a stable line arrangement of degree 5 gives a Veronese surface intersecting five given pairwise intersecting planes in conics. So, consider a stable line arrangement given by linear forms $l_0, l_1, \ldots l_4$ on $\mathbb{P}^2$. In terms of coordinates in $\mathbb{P}^2$ we may assume that $l_1 = y_0$, $l_2 = y_1$, $l_3 = y_2$, $l_4 = y_0 + y_1 + y_2$ and $l_5 = y_0 + e_1 y_1 + e_2 y_2$, where $e_1 e_2 \neq 0, e_1 \neq 1, e_2 \neq 1$ and $e_1 \neq e_2$. Given five planes P_i in $\mathbb{P}^5$ that pairwise intersect, we may choose coordinates such that the first four planes $P_1, \ldots, P_4$ are defined by $x_0 = x_1 = x_2 = 0$, $x_2 = x_3 = x_4 = 0$, $x_0 = x_4 = x_5 = 0$ and $x_1 = x_3 = x_5 = 0$, and that the fifth plane is defined by

$$x_0 + a_3 x_3 + a_4 x_4 + a_5 x_5 = 0,$$
$$x_1 + b_3 x_3 + b_4 x_4 + b_5 x_5 = 0,$$
$$x_2 + c_3 x_3 + c_4 x_4 + c_5 x_5 = 0,$$

where

$$\det \begin{pmatrix} a_3 & a_4 & a_5 \\ b_3 & b_4 & b_5 \\ c_3 & c_4 & c_5 \end{pmatrix} = 0, c_5 = 0, b_4 = 0 \quad \text{and} \quad a_3 = 0,$$

to make the fifth plane intersect the first four. This reduces to the condition that the equations of the fifth plane P_5 are

$$x_0 + a_4 x_4 + a_5 x_5 = 0,$$
$$x_1 + b_3 x_3 + b_5 x_5 = 0,$$
$$x_2 + c_3 x_3 + c_4 x_4 = 0,$$

and

$$\det \begin{pmatrix} 0 & a_4 & a_5 \\ b_3 & 0 & b_5 \\ c_3 & c_4 & 0 \end{pmatrix} = 0.$$

The last condition may be reformulated to say that there exist nonzero coefficients s_1, s_2, s_3 such that $-s_2 b_3 + s_3 c_3 = s_1 a_4 + s_3 c_4 = s_1 a_5 - s_2 b_5 = 0$. Thus we may give the equations of P_5 the form:

$$s_1 x_0 + B x_4 + C x_5 = 0,$$
$$s_2 x_1 + A x_3 + C x_5 = 0,$$
$$s_3 x_2 + A x_3 - B x_4 = 0.$$

We now define the Veronese map by a choice of basis for the quadrics on $\mathbb{P}^2$:

$$d_0 x_0 = l_1 l_3, d_1 x_1 = l_1 l_4, d_2 x_2 = l_1 l_2, d_3 x_3 = l_2 l_4, d_4 x_4 = l_2 l_3, d_5 x_5 = l_3 l_4$$

where the coefficients $d_0, \dots, d_5$ are all nonzero and to be determined. In coordinates y_i we get

$$d_0x_0 = y_0y_2, d_1x_1 = y_0(y_0 + y_1 + y_2), d_2x_2 = y_0y_1,$$

$$d_3x_3 = y_1(y_0 + y_1 + y_2), d_4x_4 = y_1y_2, d_5x_5 = y_2(y_0 + y_1 + y_2).$$

The image of $l_5 = 0$ lies in the plane defined by the three linear forms

$$e_2d_0x_0 + (d_1x_1 - d_0x_0 - d_2x_2) + e_1d_2x_2,$$

$$d_2x_2 + e_2d_4x_4 + e_1(d_3x_3 - d_2x_2 - d_4x_4),$$

and

$$d_0x_0 + e_1d_4x_4 + e_2(d_5x_5 - d_0x_0 - d_4x_4)$$

or after reordering and reduction

$$(1 - e_2)d_0x_0 + (e_1 - e_2)d_4x_4 + e_2d_5x_5$$

$$d_1x_1 + e_1d_3x_3 + e_2d_5x_5,$$

and

$$(1 - e_1)d_2x_2 + e_1d_3x_3 + (e_2 - e_1)d_4x_4.$$

So this plane coincides with P_5 when

$(1 - e_2)d_0 = s_1, d_1 = s_2, (1 - e_1)d_2 = s_3, e_1d_3 = A, (e_1 - e_2)d_4 = B, e_2d_5 = C,$ which determines the coefficients $d_0, \dots, d_5$ as promised. An immediate consequence of this computation is

Proposition 2.4. *The surface Y is rational.*

We shall now describe a compactification Y of Y_2, that we shall eventually show coincides with the Hilbert scheme compactification Y_2^H. For each Veronese surface the union of secant lines form a singular cubic hypersurface. The cubic hypersurface is defined by the determinant of a symmetric 3×3 matrix of linear forms, and it is singular precisely along the Veronese surfaces. Therefore Y_2 as a reduced variety has a natural embedding in the space of cubic hypersurfaces. We take Y to be the reduced closure of Y_2 in this embedding. Our main result in this section is the description of Y.

Theorem 2.5. *The compactification Y of the family Y_2 of Veronese surfaces in $\mathbb{P}^5$ that meet five pairwise intersecting planes in conics, form a smooth Del Pezzo surface of degree 5. The lines on the Del Pezzo surface form the boundary $Y \setminus Y_2$ and parameterizes degenerate Veronese surfaces. A point that lies on one line corresponds to a surface with a smooth cubic scroll component and a plane through the directrix of the scroll, while the intersection point between two lines corresponds to a surface with a smooth quadric component and two plane components meeting each other in a point and the quadric in a line, one from each ruling.*

Remark 2.6. *Kapranov [2] proves that by association, the moduli space Y_2 is isomorphic to the moduli space $M_{0,5}$ of 5-pointed $\mathbb{P}^1$ which is isomorphic to a Del Pezzo surface of degree 5.*

Proof. Consider the family of cubic hypersurfaces singular along a Veronese surface of the family Y_2. Their linear span is contained in the vector space V of cubic hypersurfaces that are singular at the ten points of intersection of the five planes P_i. This space V has dimension at least 6: The space of cubics that contain the five pairwise intersecting planes is 16-dimensional. A singularity at an intersection point imposes one linear condition, so altogether the singularities imposes at most ten conditions, which leaves at least a 6-dimensional vector space for the 10-nodal cubics.

Lemma 2.7. *No cubic in V is singular along one of the five planes P_i.*

Proof. Let F be a cubic singular along the plane P_1, say. If so let P' be a $3-$space spanned by P_1 and the point of intersection between two of the other planes. Since F is singular in the intersection point, the span P' must be contained in F. Consider the hyperplane H_{12} spanned by P_1 and P_2. By the previous argument the restriction of F to H_{12} decomposes into three $\mathbb{P}^3$'s that contain P_1, one for each intersection point of P_2 with the remaining three planes. On the other hand F contains P_2, so H_{12} must be a component of F. Similarly the hyperplanes spanned by P_1 and P_3, P_4 and P_5 are all components of F. But F is a cubic, so this is absurd. □

Consequently, there is precisely a net of cubics in V that are singular at some point in P_i outside the original four. In fact since the cubics already contain P_i, it is at most 3 conditions to be singular at some point in the plane, and if it is less than 6 conditions to be singular at two general points outside the original four, then there would be a cubic hypersurface singular along the whole plane, contradicting Lemma 2.7. In particular

Corollary 2.8. *The space $\mathbb{P}(V)$ of cubic hypersurfaces that are singular in the ten intersection points of the five planes P_i is $5-$dimensional and the cubic hypersurfaces that are singular at some point in P_i outside the original four form a $\mathbb{P}^2$ and are all singular along the unique conic passing through the five points.*

Let $V_i \subset \mathbb{P}(V)$ be the subvariety of cubic hypersurfaces that are singular along some conic in P_i.

Lemma 2.9. *The variety V_i is a rational cubic scroll, the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^2$.*

Proof. By Corollary 2.8 there is a plane in V_i of cubic hypersurfaces for each conic. Since the conics moves in a pencil, the variety V_i is the union of a pencil of planes. Furthermore, no two planes in this pencil have a common point, because the corresponding cubic hypersurface would be singular along P_i contradicting Lemma 2.7. But the only pencils of planes with this property are those of the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^2$, so V_i must be such a cubic scroll. □

The five cubic scrolls V_i are distinct: Let H_{ij} be the hyperplane spanned by P_i an P_j, let p_{ij} be the intersection point $P_i \cap P_j$, and let $H_i(jk, lm)$ be the hyperplane spanned by P_i and the points p_{jk} and p_{lm}. Then $H_{25} \cup H_{34} \cup H_5(12, 34)$ and $H_{12} \cup H_{34} \cup H_1(25, 34)$ form cubic hypersurfaces that belong to $\mathbb{P}(V)$. They are both singular along the conic $(H_{25} \cup H_{34}) \cap P_1$ in P_1, but they are singular at the distinct conics $(H_{25} \cup H_{13}) \cap P_4$ and $(H_{35} \cup H_{12}) \cap P_4$ in P_4. Since the planes in V_i correspond to conics in P_i, the planes in V_1 and V_4 say, cannot coincide. Therefore, the V_i are also distinct.

Now, a cubic scroll is defined by the 2×2 minors of a 2×3 matrix of linear forms, i.e. be three quadrics. Since the intersection of two of these quadrics is a threefold of degree 4, two distinct cubic scrolls have at most one defining quadrics in common. Therefore the ideal of Y contains at least 5 quadrics, and Y is contained in five distinct cubic scrolls. In particular, Y has degree at most 5.

Next, we will enumerate degenerations of Veronese surfaces in V_2. So we consider limit schemes T in the Hilbert scheme of the Veronese surfaces in our family. A limit scheme T must clearly satisfy the following two conditions:

1. The Hilbert polynomial of T coincides with that of a smooth Veronese surface. In particular it has a surface component of degree 4 whose reduced part is connected in codimension 1.

2. The scheme theoretic intersection with any one of the five planes is at least a plane conic that passes through the four intersection points of this plane with the other planes.

First, we need some notation. As above we denote by p_{ij} the intersection point $P_i \cap P_j$, and by H_{ij} the hyperplane spanned by P_i and P_j. Denote by P_{ijk} the plane spanned by the interesection points of P_i, P_j and P_k, and let L_{ijk} be the line of intersection $P_{ijk} \cap H_{lm}$.

To enumerate the possible degenerate Veronese surfaces, we first exclude two cases. First, one of the planes P_i cannot be a component of a degenerate Veronese surface: If so, the residual surface S has degree 3 and intersect the four remaining planes in at least a conic. Any two of these four planes span a hyperplane that intersect S in a curve of degree at least 4, so S has a component in each of these 6 hyperplanes. But any three of these hyperplanes intersect in a plane, therefore S has at least 4 components. This is a contradiction. Secondly, any quadric surface component cannot intersect one of the five planes, say P_5 in a conic: If so this component intersects the remaining four planes in at most two lines, in which case these lines meet at an intersection point of two planes, say p_{34}. The residual surface S has degree 2 and intersect P_1 and P_2 in at least a conic, and P_3 and P_4 in at least a line. Again, this means that the hyperplanes spanned by H_{12}, H_{13}, H_{14}, H_{23} and H_{24} all contain a component of S. This is possible only if S consists of the two planes P_{123} and P_{124}. But then the intersection of S and the quadric is 0-dimensional, against condition 1 above.

Now, any component of a degenerate Veronese surface meets a plane P_i in a conic section, a line, a point or not at all. Consider components of reducible degenerate Veronese surfaces. By the first condition, they are planes, quadrics or cubic scrolls. Since at most two planes P_i lie in a $\mathbb{P}^4$ any cubic scroll intersects the union of the P_i in a curve of degree at most 7. In that case it intersect two planes in a conic, and the remaining three in a line. Furthermore these three lines are disjoint, so the scroll is smooth. Since a $\mathbb{P}^3$ intersect at most 4 planes in a line, and no quadric component can intersect a plane in a conic, the intersection between a quadric component and the five planes is at most 4 lines. In that case they form a quadrangle defined by the pairwise intersection of two pairs of planes and the quadric surface component is smooth. Finally, any plane intersect the five planes in at most three lines, in which case it is the plane spanned by the intersection points of three planes.

Lemma 2.10. *If T is a singular Veronese surface in the family Y, then T is the reducible union of a plane and a smooth cubic scroll intersecting the plane along* *the directrix, or the reducible union of two planes and a smooth quadric surface intersecting the planes along a line one from each ruling.*

Proof. First, an irreducible singular degeneration of a Veronese surface is a cone over a rational normal curve. Since the intersection with each plane is a conic, the vertex of the cone is in each plane, absurd. A nonreduced component is a multiple structure on a plane or a double structure on a quadric. The bound of the intersection of the reduced component with the five planes excludes this possibility also. Therefore a singular Veronese surface in the family is reduced, and the components are planes, smooth quadrics or cubic scrolls. Furthermore, by degree reasons alone, any plane component must meet the union of the five planes in 3 lines, any quadric surface component is smooth and must meet the five planes in 4 lines. Furthermore if the singular Veronese surface has a quadric surface component, the residual components must be two planes, that meet the quadric in a line, one from each ruling. Finally, any cubic scroll intersect the five planes in a curve of degree 7 with two conic components and 3 disjoint lines. In particular the scroll is smooth, and the residual component is a plane through the directrix. □

We may now enumerate the degenerate Veronese surfaces in the above notation:

1. Any plane component coincides with the plane spanned by the intersection points of three planes $P_{ijk} =< p_{ij}, p_{jk}, p_{ik} >$.

2. Any quadric component is smooth and coincides with the quadric through say L_{123} and L_{345} which contain the points $p_{14}, p_{24}, p_{15}, p_{25}$. This quadric lies in the 3-space $< P_1, P_2 > \cap < P_4, P_5 >$.

3. Any cubic scroll component is smooth and coincides with a scroll through three lines and meeting two planes in conics through their intersection point. For each pair of planes there is a pencil of them, i.e. for each plane component there is a pencil of them. For the plane P_{ijk} this pencil of cubic scrolls have the line L_{ijk} as a common directrix.

Altogether we find ten plane components in degenerations, fifteen smooth quadric surface components of degenerations, and ten pencils of smooth cubic scroll components.

For each of these degenerate Veronese surfaces the union of secant lines form a cubic fourfold: If the Veronese surface is smooth, then the cubic is the determinant of a symmetric matrix. If the surface is the union of a plane and a cubic scroll, then the cubic is the union of the $\mathbb{P}^4$ of the cubic scroll and the unique rank 3 quadric with vertex the plane which contains the cubic scroll. If the surface is the union of two planes and a quadric surface, then the cubic is the union of three $\mathbb{P}^4$'s. Notice furthermore that in each case the union of the secant lines form the unique cubic hypersurface that is singular along the degenerate Veronese surface.

Therefore there is a curve consisting of ten lines in $\mathbb{P}(V)$ parameterising singular cubics of degenerate Veronese surfaces.

Lemma 2.11. *The surface $Y \subset \mathbb{P}(V)$ contains at most ten lines.*

Proof. Assume that a line in Y contains the cubic of a smooth Veronese surface, and consider the base locus of the pencil. Since every cubic of the pencil is singular along some Veronese surface, by Bertini, the union of the Veronese surfaces must be contained in the base locus of the pencil. In at least one of the planes P_i the pencil of Veronese surfaces restricts to the pencil of conics through the four intersection points, since otherwise the pencil of Veronese surfaces have five conics in common, which is impossible. Corresponding to the singular conics in the pencils, the Veronese must be degenerate, so the cubic is decompose in a hyperplane and a rank 3 quadric with vertex a plane P_{ijk}, or in three hyperplanes. If the base locus contains a Veronese surface, this must be contained in the rank 3 quadric and the vertex plane is a tangent plane to the Veronese surface. For each singular conic there is a rank 3 quadric containing the Veronese surface, such that the vertex plane is tangent to the Veronese surface. But two of these three quadrics must have a vertex plane that passes through the same intersection point p_{ij} on P_i. On the other hand this is a point on the Veronese surface, so in fact it must be the point of tangency, and the two vertex planes must coincide, which is absurd. Therefore all cubics of the pencil are cubics of degenerate Veronese surfaces, and the ten lines of degenerate Veronese surfaces are the only possible lines on Y. □

Clearly, both Y and the ten lines of degenerate Veronese surfaces lie in the five cubic scrolls V_i. Recall that Y has degree at most 5. On the other hand Y spans $\mathbb{P}(V)$: Otherwise it is a possibly reducible cubic scroll, but Y contains contains at most ten lines. If Y has degree 4, it is again a scroll or a possibly degenerate Veronese surface. Only in the case of a Veronese surface are there at most ten lines. But a Veronese surface is not contained in any smooth cubic threefold scroll, so we are left with the sole possibility that Y has degree 5 and is contained in 5 quadrics. This is possible only if the ideal of Y is generated by five quadrics, the Pfaffians of a 5×5 matrix of linear forms. Thus Y is the scheme theoretic intersection of the cubic scrolls, the ten lines all lie on Y and Y is a smooth Del Pezzo surface. □

Corollary 2.12. *Y coincides with the Hilbert scheme compactification Y_2^H of Y_2.*

Proof. The family Y of surfaces is clearly flat, so there is a morphism $f : Y \to Y_2^H$. On the other hand the singular cubic hypersurfaces associated to each surface in Y sets up a morphism $g : Y_2^H \to Y$ inverse to f, so f must be an isomorphism. □

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Department of Mathematics, University of California, Berkeley, CA 94720
E-mail address: graber@math.berkeley.edu

Matematisk Institutt, Universitetet i Oslo P.O.Box 1053, Blindern, N-0316 Oslo, NORWAY
E-mail address: ranestad@math.uio.no