REAL k-FLATS TANGENT TO QUADRICS IN $\mathbb{R}^n$

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Abstract. Let $d_{k,n}$ and $\#_{k,n}$ denote the dimension and the degree of the Grassmannian $\mathbb{G}_{k,n}$, respectively. For each $1 \leq k \leq n-2$ there are $2^{d_{k,n}} \cdot \#_{k,n}$ (a priori complex) k-planes in $\mathbb{P}^n$ tangent to $d_{k,n}$ general quadratic hypersurfaces in $\mathbb{P}^n$. We show that this class of enumerative problem is fully real, i.e., for $1 \leq k \leq n-2$ there exists a configuration of $d_{k,n}$ real quadrics in (affine) real space $\mathbb{R}^n$ so that all the mutually tangent k-flats are real.

Introduction

Understanding the real solutions of a system of polynomial equations is a fundamental problem in mathematics (see, e.g., [13] for some recent lines of research and applications). However, as pointed out in [3, p. 55], even for problem classes with a finite number of complex solutions (enumerative problems), the question of how many solutions can be real is still widely open. A class of enumerative problems is called *fully real* if there are general real instances for which all the (a priori complex) solutions are real.

One of us (Sottile) began a systematic study of this question in the *special Schubert calculus* [9, 10], a class of enumerative problems from classical algebraic geometry. This special Schubert calculus asks for linear subspaces of a fixed dimension meeting some given (general) linear subspaces (whose dimensions and number ensure a finite number of solutions) in n-dimensional complex projective space $\mathbb{P}^n$. For any given dimensions of the subspaces, this problem is fully real, i.e., there exist *real* linear subspaces for which each of the a priori complex solutions are *real*. In particular, for $1 \leq k \leq n-2$ there are $d_{k,n} := (k+1)(n-k)$ real $(n-k-1)$-planes $U_1, \ldots, U_{d_{k,n}}$ in $\mathbb{P}^n$ with

$$\#_{k,n} := \frac{1!2!\cdots k!((k+1)(n-k))!}{(n-k)!(n-k+1)!\cdots n!}$$

real k-planes meeting $U_1, \ldots, U_{d_{k,n}}$. Here, $d_{k,n}$ and $\#_{k,n}$ are the dimension and the degree of the Grassmannian $\mathbb{G}_{k,n}$, respectively (see [5, 7]). These were the first results showing that a large class of non-trivial enumerative problems is fully real.

We continue this line of research by considering k-flats tangent to quadratic hypersurfaces (hereafter *quadrics*). This is also motivated by recent investigations in computational geometry (see [6, 11, 12]). It was shown in [12] that $2n-2$ general *spheres* in affine real space $\mathbb{R}^n$ have at most $3 \cdot 2^{n-1}$ common tangent lines in $\mathbb{C}^n$,

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and that there exist spheres for which all the a priori complex tangent lines are *real*. The present paper addresses the following question: What is the maximum number of real k-flats simultaneously tangent to $d_{k,n}$ general quadrics in $\mathbb{R}^n$ (respectively in $\mathbb{P}^n_{\mathbb{R}}$)? As this problem may be formulated as the complete intersection of $d_{k,n}$ quadratic equations on the Grassmannian of k-planes in $\mathbb{P}^n$, the expected number of complex solutions is the product of the degrees of the equations with the degree of the Grassmannian, i.e., $2^{d_{k,n}} \cdot \#_{k,n}$. We show that the problem is fully real:

Theorem 1. *Let $1 \le k \le n-2$. Given $d_{k,n}$ general quadrics in $\mathbb{P}^n$ there are $2^{d_{k,n}} \cdot \#_{k,n}$ complex k-planes that are simultaneously tangent to all $d_{k,n}$ quadrics. Furthermore, there is a choice of quadrics in $\mathbb{R}^n$ for which all the k-flats are real, distinct, and lie in affine space $\mathbb{R}^n$.*

When $k = 1$, we have $d_{1,n} = 2(n-1)$ and $\#_{1,n}$ is the Catalan number $\#_{1,n} = \frac{1}{n}\binom{2n-2}{n-1}$. The following table exhibits the large discrepancy between the number of lines tangent to spheres and the number of lines tangent to general quadrics. When $n = 3$ this discrepancy was accounted for by Aluffi and Fulton [1].

n	3	4	5	6	7	8	9
$3 \cdot 2^{n-1}$	12	24	48	96	192	384	768
$2^{d_{1,n}} \cdot \#_{1,n}$	32	320	3584	43008	540672	7028736	93716480

In Section 1, we review some facts on Plücker coordinates of k-planes in projective space. In Section 2, we combine recent results in the real Schubert calculus with classical perturbation arguments adapted to the real numbers to prove Theorem 1. Since the proof for general (k, n) is non-constructive, we give a symbolic, constructive proof for the case $(k, n) = (1, 3)$ in Section 3.

1. Preliminaries

We review the well-known *Plücker coordinates* of k-dimensional linear subspaces (hereafter k-planes) in complex projective space $\mathbb{P}^n$ (see, e.g., [4]). Let U be a k-plane in $\mathbb{P}^n$ which is spanned by the columns of an $(n+1) \times (k+1)$-matrix L. For every subset $I \subset \{0, \ldots, n\}$ of size $k+1$ let p_I be the $(k+1) \times (k+1)$-subdeterminant of L given by the rows in I and let $N := \binom{n+1}{k+1} - 1$. Then $p := (p_I)_{I \subset \{0,\ldots,n\}, |I|=k+1} \in \mathbb{P}^N$ is the *Plücker coordinate* of U. The set of all k-planes in $\mathbb{P}^n$ is called the *Grassmannian of k-planes in* $\mathbb{P}^n$ and is denoted by $\mathbb{G}_{k,n}$. If the indices are written as ordered tuples then the Plücker coordinates are skew-symmetric in the indices. $\mathbb{G}_{k,n}$ is in 1-1-correspondence with the set of vectors in $\mathbb{P}^N$ satisfying the *Plücker relations*, i.e.,

$$\sum_{l=1}^{k+1} (-1)^l p_{i_1 \ldots \hat{i}_l \ldots i_{k+1}} \, p_{j_1 \ldots j_{k-1} i_l} \;=\; 0 \qquad (1)$$

for every $I = \{i_1, \ldots, i_{k+1}\}$, $J = \{j_1, \ldots, j_{k-1}\} \subset \{0, \ldots, n\}$ of strictly ordered index sets (where $\hat{}$ over an index means that it is omitted). See, e.g., [4, §VII.6]. By Schubert's results [7], the dimension of $\mathbb{G}_{k,n}$ is $d_{k,n} = (k+1)(n-k)$ and its degree is $\#_{k,n}$.

If an $(n-k-1)$-plane V is given as the intersection of the $k+1$ hyperplanes $\sum_{i=0}^{n} v_i^{(0)} x_i = 0, \ldots, \sum_{i=0}^{n} v_i^{(k)} x_i = 0$, then the *dual Plücker coordinate* $q = (q_I)_{I\subset\{0,\ldots,n\},|I|=k+1} \in \mathbb{P}^N$ of V is defined by the $(k+1)\times(k+1)$-subdeterminants of the matrix with columns $v^{(0)}, \ldots, v^{(k)}$.

A k-plane U intersects an $(n-k-1)$-plane V in $\mathbb{P}^n$ if and only if the dot product of the Plücker coordinate p of U and the dual Plücker coordinate q of V vanishes, i.e., if and only if

$$\sum_{I\subset\{0,\ldots,n\},|I|=k+1} p_I q_I \;=\; 0 \tag{2}$$

(see, e.g., [4, Theorem VII.5.I]).

We use Plücker coordinates to characterize the k-planes tangent to a given quadric in $\mathbb{P}^n$ (see [11]). We identify a quadric $x^{\mathrm{T}}Qx = 0$ in $\mathbb{P}^n$ with its symmetric $(n+1)\times(n+1)$-representation matrix Q. Further, for $r \in \mathbb{N}$ let $\wedge^r$ denote the r-th exterior power of matrices

$$\wedge^r \;:\; \mathbb{C}^{m\times n} \;\;\to\;\; \mathbb{C}^{\binom{m}{2}\times\binom{n}{2}}$$

(see [11]). The row and column indices of the resulting matrix are subsets of cardinality r of $\{1,\ldots,m\}$ and $\{1,\ldots,n\}$, respectively. For $I \subset \{1,\ldots,m\}$ with $|I| = r$ and $J \subset \{1,\ldots,n\}$ with $|J| = r$, $\left(\wedge^r A\right)_{I,J}$ is the subdeterminant of A whose rows are indexed by I and whose columns are indexed by J. If a k-plane $U \subset \mathbb{P}^n$ is spanned by the columns of an $(n+1)\times(k+1)$-matrix L, then the $\binom{n+1}{k+1}\times 1$-matrix $\wedge^{k+1}L$, considered as a vector in $\mathbb{P}^N$, is the Plücker coordinate of U.

Recall the following algebraic characterization of tangency: A k-plane U is tangent to a quadric Q if the restriction of the quadratic form to U is singular (which includes the case $U \subset Q$). When the quadric is smooth, this implies that U is tangent to the quadric in the usual geometric sense.

Proposition 2 (Proposition 5.5.3 of [11])**.** *A k-plane $U \subset \mathbb{P}^n$ is tangent to a quadric Q if and only if the Plücker coordinate p_U of U satisfies*

$$p_U^{\mathrm{T}}\left(\wedge^{k+1}Q\right)p_U = 0\,. \tag{3}$$

A *k-flat* in affine real space $\mathbb{R}^n$ is a k-dimensional affine subspace in $\mathbb{R}^n$. Throughout the paper we assume that $\mathbb{R}^n$ is naturally embedded in the real projective space $\mathbb{P}^n_{\mathbb{R}}$ via $(x_1,\ldots,x_n) \mapsto (1,x_1,\ldots,x_n) \in \mathbb{P}^n_{\mathbb{R}}$.

2. Proof of the main theorem

We first illustrate the essential geometric idea underlying our constructions for $(k,n) = (1,3)$, which is the first nontrivial case. Here, Theorem 1 states that there exists a configuration of four quadrics in $\mathbb{R}^3$ with 32 distinct real common tangent lines.

By (2), the set of lines meeting four given lines in $\mathbb{P}^3$ is the intersection of four hyperplanes on the Grassmannian $\mathbb{G}_{1,3}$, and hence there are at most two or infinitely many common lines meeting $\ell_1,\ldots,\ell_4$. If e_1 and e_2 are opposite edges in a tetrahedron

$\Delta \subset \mathbb{R}^3$, then the lines underlying e_1 and e_2 are the two common transversals of the four lines underlying the other four edges (see Figure 1).

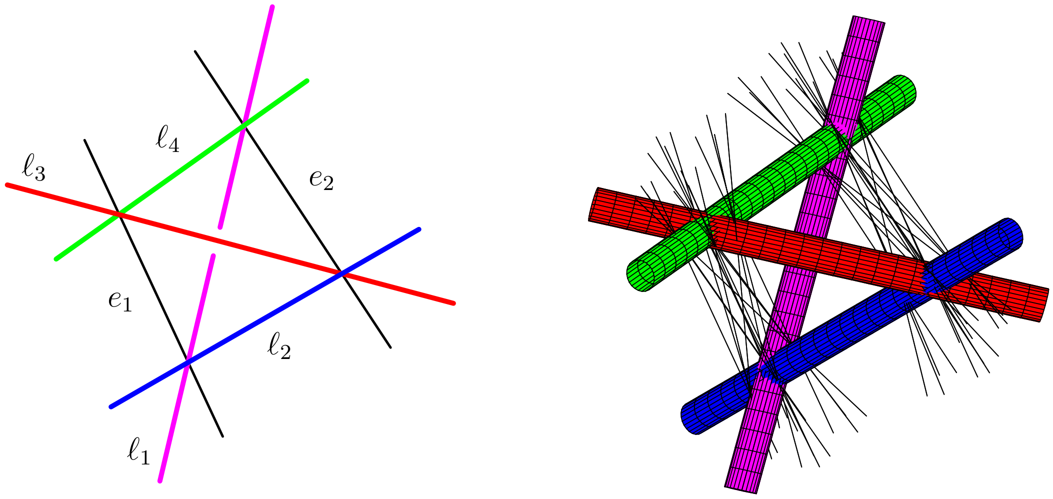

FIGURE 1. A tetrahedron configuration of four lines in $\mathbb{R}^3$ with two real transversals and a configuration of four quadrics with 32 real tangents.

Consider the lines $\ell_1, \ldots, \ell_4$ as (degenerate) infinite circular cylinders with radius $r = 0$. When the radius is slightly increased, then the cylinders intersect pairwise in the regions (combinatorially) given by the four vertices of Δ, and the common tangents roughly have the direction of e_1 or e_2. Since the neighborhood of a vertex is divided into four regions by the two cylinders, and since each region contains common tangents, this gives $4 \cdot 4$ tangents close to the direction of e_1 and $4 \cdot 4$ tangents close to the direction of e_2 (see Figure 1.)

For the general case, let $1 \leq k \leq n-2$. By Section 1, the number of k-planes in $\mathbb{P}^n$ simultaneously meeting $d_{k,n}$ general $(n-k-1)$-planes is $\#_{k,n}$. We begin with a configuration of $d_{k,n}$ real $(n-k-1)$-flats $U_1, \ldots, U_{d_{k,n}}$ in $\mathbb{R}^n$ having $\#_{k,n}$ real $(n-k-1)$-flats simultaneously meeting $U_1, \ldots, U_{d_{k,n}}$. We then argue that we can replace each of these $(n-k-1)$-flats by a real quadric such that for each of the k-flats, there are $2^{d_{k,n}}$ nearby real k-flats tangent to each quadric.

Proposition 3. *For $1 \leq k \leq n-2$, there exists a configuration of $d_{k,n}$ real $(n-k-1)$-flats $U_1, \ldots, U_{d_{k,n}}$ in $\mathbb{R}^n$ such that there exist exactly $\#_{k,n}$ real k-flats simultaneously meeting $U_1, \ldots, U_{d_{k,n}}$.*

Proof. The corresponding statement for real projective space $\mathbb{P}^n_{\mathbb{R}}$ was proven for $k = 1$ in [9, Theorem C] and for $k \geq 2$ in [10]. We deduce the affine counterpart above simply by removing a real hyperplane that contains none of the $(n-k-1)$-flats or any of the transversal k-flats. $\square$

For $k = 1$, the purely existential statement in [9] and Proposition 3 was improved by Eremenko and Gabrielov [2] who gave the following explicit construction of such a collection of $(n-2)$-flats. Let $\gamma : \mathbb{R} \to \mathbb{R}^n$, $\gamma(s) := (1, s, s^2, \ldots, s^{n-1})^{\mathrm{T}}$ be the moment curve in $\mathbb{R}^n$. For each $s \in \mathbb{R}$, set $U(s)$ to be

$$U(s) := \text{linear span}\big(\gamma(s), \gamma'(s), \ldots, \gamma^{(n-3)}(s)\big) .$$

Geometrically, $U(s)$ is the $(n-2)$-flat osculating the moment curve at the point $\gamma(s)$. By [2], for any distinct $s_1, \ldots, s_{2n-2} \in \mathbb{R}$, the $(n-2)$-flats $U(s_1), U(s_2), \ldots, U(s_{2n-2})$ have exactly $\#_{1,n} = C_{n-1}$ common real transversals, where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the n-th Catalan number. For general k, it is only known that there *exist* distinct $s_1, \ldots, s_{d_{k,n}} \in \mathbb{R}$ such that there are $\#_{k,n}$ distinct real k-flats meeting the osculating $(n-k-1)$-flats to the moment curve at $s_1, \ldots, s_{d_{k,n}}$ [10]. The *conjecture on total reality* in [8, §1 and §4] conjectures that any choice of distinct $s_1, \ldots, s_{d_{k,n}} \in \mathbb{R}$ implies reality of all transversal subspaces.

Definition. Suppose that $1 \le k \le n-2$, and let $U \subset \mathbb{R}^n$ be a k-flat and $r > 0$. The *k-cylinder* $\mathrm{Cy}(U, r)$ is the set of points having Euclidean distance r from U.

This quadratic hypersurface is smooth in $\mathbb{R}^n$ but its extension to $\mathbb{P}^n$ is singular. A k'-flat $V \subset \mathbb{R}^n$ is tangent to $\mathrm{Cy}(U, r)$ if and only if its Euclidean distance to U is r.

We will use the following basic property of intersection multiplicities [3, p. 1].

Proposition 4. *Let $\mathcal{A}$ be an algebraic curve in $\mathbb{P}^n$, and let x be a singular point on $\mathcal{A}$. For any hyperplane $H \subset \mathbb{P}^n$ such that x is an isolated point in $\mathcal{A} \cap H$, the intersection multiplicity of $\mathcal{A}$ and H in x is greater than* 1.

Theorem 5. *Let $1 \le k \le n-2$, and let $U_1, U_2, \ldots, U_{d_{k,n}}$ be $(n-k-1)$-flats in $\mathbb{R}^n$ having exactly $\#_{k,n}$ common transversal k-flats, all real. For each $i = 0, 1, \ldots, d_{k,n}$, there exist $r_1, \ldots, r_i > 0$ such that there are exactly $2^i \cdot \#_{k,n}$ distinct k-flats, each of them real, that are simultaneously tangent to each of the $(n-k-1)$-cylinders* $\mathrm{Cy}(U_j, r_j)$, *$j = 1, \ldots, i$, and also meet each of the $(n-k-1)$-flats $U_{i+1}, \ldots, U_{d_{k,n}}$.*

The case of $i = d_{k,n}$ implies Theorem 1.

Proof. We induct on i, with the case of $i = 0$ being the hypothesis of the theorem.

Suppose that $i \le d_{k,n}$ and that there exist $r_1, \ldots, r_{i-1} > 0$ such that there are exactly $2^{i-1} \cdot \#_{k,n}$ distinct k-flats $V_1, \ldots, V_{2^{i-1}\#_{k,n}}$ which are simultaneously tangent to $\mathrm{Cy}(U_j, r_j)$ for each $j = 1, \ldots, i-1$, and meet each of $U_i, \ldots, U_{d_{k,n}}$, and each of these k-flats is real.

Now we drop the condition that the k-flats meet U_i. Let $\mathcal{A} \subset \mathbb{G}_{k,n}$ be the curve of k-flats that are tangent to the cylinders $\mathrm{Cy}(U_j, r_j)$ for $j = 1, \ldots, i-1$ and that also meet each of the $(n-k-1)$-flats $U_{i+1}, \ldots, U_{d_{k,n}}$. Since $\mathcal{A}$ is the intersection of $i-1$ quadrics (the tangency conditions) with $d_{k,n}-i$ hyperplanes (conditions to meet the remaining U_j) on the Grassmannian, it has degree at most $2^{i-1}d_{k,n}$. Since its intersection with the hyperplane defined by U_i consists of $2^{i-1}d_{k,n}$ points, we conclude that the degree of $\mathcal{A}$ is $2^{i-1}d_{k,n}$ and (by Proposition 4) that each of these points is a smooth point of $\mathcal{A}$.

Let $V \in \{V_1, \ldots, V_{2^{i-1}\#_{k,n}}\}$. Since V is a smooth real point of the real curve $\mathcal{A} \subset \mathbb{G}_{k,n}$ (i.e., $V \subset \mathbb{P}^n_{\mathbb{R}}$), the real points of $\mathcal{A}$ contain a smooth arc α containing V with $\alpha \cap (\{V_1, \ldots, V_{2^{i-1}\#_{k,n}}\} \setminus V) = \emptyset$. Let $\varphi\colon (-\delta, \delta) \to \alpha$ be a smooth parametrization of the arc α with $\varphi(0) = V$. Such a parametrization exists, for example, by the Implicit Function Theorem.

Thus, for $t \in (-\delta, \delta) \setminus \{0\}$, the real k-flat $\varphi(t)$ does not meet U_i and so it has a positive Euclidean distance $d(t)$ from U_i. Since $d(t)$ is a continuous function of t, for $\rho \in \mathbb{R}$ with $0 < \rho < \min\{d(-\delta/2), d(\delta/2)\}$ there are at least two distinct real k-flats in α whose Euclidean distance to U_i is ρ.

In this way, we obtain $2^{i-1} \cdot \#_{k,n}$ such arcs, each containing one of $V_1, \ldots, V_{2^{i-1}\#_{k,n}}$. We may assume that these arcs are pairwise disjoint. Let $0 < r_i$ be small enough to ensure that each arc contains two k-flats having Euclidean distance r_i from U_i. This gives *at least* $2 \cdot 2^{i-1} \cdot \#_{k,n}$ real k-planes in $\mathcal{A}$ whose Euclidean distance to U_i is r_i. Since $2^i \cdot \#_{k,n}$ is the maximum number of k-flats with this property, there are exactly $2^i \cdot \#_{k,n}$ distinct k-flats tangent to $\mathrm{Cy}(U_j, r_j)$ for $j = 1, \ldots, i$ and that also meet each of the $(n-k-1)$-flats $U_{i+1}, \ldots, U_{d_{k,n}}$. □

Since the number of real k-flats will not change under a small perturbation of the k-cylinders $\mathrm{Cy}(U_j, r_j)$, we may replace them by quadrics which are smooth in $\mathbb{P}^n$. Let $\mathrm{sign}(Q)$ denote the signature of a quadric $Q \subset \mathbb{P}^n$.

Corollary 6. *Let $1 \leq k \leq n-2$. For*

$$(s_1, \ldots, s_{d_{k,n}})^{\mathrm{T}} \in \begin{cases} \{n-1, n-3, \ldots, 2k-n+1\}^{d_{k,n}} & \text{if } k \geq n/2\,, \\ \{n-1, n-3, \ldots, 2 \cdot \left(\frac{n-1}{2} - \lfloor \frac{n-1}{2} \rfloor\right)\}^{d_{k,n}} & \text{if } k < n/2 \end{cases}$$

there exist smooth quadrics $Q_1, \ldots, Q_{d_{k,n}} \subset \mathbb{P}^n_{\mathbb{R}}$ with $|\mathrm{sign}(Q_i)| = s_i$, $1 \leq i \leq d_{k,n}$, such that the $\#_{k,n}$ (complex) common tangent k-flats to $Q_1, \ldots, Q_{d_{k,n}}$ are all real, distinct, and lie in affine space $\mathbb{R}^n$.

Proof. Since the signature of an $(n-k-1)$-cylinder is k, the proof immediately follows from the possible perturbations of the quadratic form in $\mathbb{P}^n$ of the type

$$-r^2 x_0^2 + x_1^2 + \cdots + x_{k+1}^2\,.$$

□

We conjecture that the reality statement holds for signatures not covered by Corollary 6.

Conjecture 7. *Let $1 \leq k \leq n-2$. For*

$$(s_1, \ldots, s_{d_{k,n}})^{\mathrm{T}} \in \{n-1, n-3, \ldots, 2 \cdot \left(\tfrac{n-1}{2} - \lfloor \tfrac{n-1}{2} \rfloor\right)\}^{d_{k,n}}$$

there exist smooth quadrics $Q_1, \ldots, Q_{d_{k,n}} \subset \mathbb{P}^n_{\mathbb{R}}$ with $|\mathrm{sign}(Q_i)| = s_i$, $1 \leq i \leq d_{k,n}$, such that the $\#_{k,n}$ (complex) common tangent k-flats to $Q_1, \ldots, Q_{d_{k,n}}$ are all real, distinct, and lie in affine space $\mathbb{R}^n$.

The first case of this conjecture which is not covered by Corollary 6 is when $k = 3$ and $n = 5$ and the signature is zero. That is, for 3-flats tangent to 8 smooth quadrics in $\mathbb{P}^5_{\mathbb{R}}$, with at least one having signature zero. We remark that an argument perturbing cylinders to singular quadrics gives an analog to Corollary 6 concerning k-flats tangent to singular quadrics. We omit its complicated formulation.

3. A constructive proof for lines in dimension 3

Our proof of Theorem 1 was non-constructive. We close this paper by providing a constructive proof in the first nontrivial case, $(k,n) = (1,3)$, i.e., the real lines tangent to four quadrics in 3-space. In order to realize the tetrahedral configuration of Figure 1 in $\mathbb{P}^3_{\mathbb{R}}$, let $\ell_1,\ldots,\ell_4$ be given by the following equations:

(4)
$$\ell_1 : x_0 = x_3 = 0\,; \quad \ell_2 : x_0 = x_1 = 0\,; \quad \ell_3 : x_1 = x_2 = 0\,; \quad \ell_4 : x_2 = x_3 = 0\,.$$

The two common transversal lines are given by $x_2 = x_4 = 0$ and by $x_1 = x_4 = 0$.

For parameters $\alpha, \beta \in \mathbb{R}$, consider the four quadrics

$$\begin{array}{rcrcl} Q_1 & : & x_0^2 + x_3^2 - \beta(x_1^2 + x_2^2) & = & 0\,, \\ Q_2 & : & x_0^2 + x_1^2 - \beta(x_2^2 + x_3^2) & = & 0\,, \\ Q_3 & : & x_1^2 + x_2^2 - \alpha(x_0^2 + x_3^2) & = & 0\,, \\ Q_4 & : & x_2^2 + x_3^2 - \alpha(x_0^2 + x_1^2) & = & 0\,. \end{array}$$

For $\alpha = \beta = 0$, the four quadrics become the corresponding lines in $\mathbb{P}^3_{\mathbb{R}}$. For small $\alpha, \beta > 0$, these quadrics are deformations of the lines with rank 4 and signature 0—smooth ruled surfaces.

Theorem 8. *Let $(\alpha,\beta) \in \mathbb{R}^2$ satisfy*

$$\alpha\beta(1-\alpha\beta)(1-\beta^2)(1-\alpha^2)\big((1-\alpha)^2(1-\beta)^2 - 16\alpha\beta\big) \neq 0\,.$$

Then there are 32 distinct (possibly complex) common tangent lines to $Q_1,\ldots,Q_4$. If $0 < \alpha, \beta < 3 - 2\sqrt{2}$, then each of these 32 tangent lines are real.

Proof. Since the quadrics only contain monomials of the form x_i^2, the tangent equations (3) of $Q_1,\ldots,Q_4$ only contain monomials of the form p_{ij}^2. Hence, the four tangent equations give the following system of linear equations in $p_{01}^2,\ldots,p_{23}^2$:

$$\begin{pmatrix} -\beta & -\beta & 1 & \beta^2 & -\beta & -\beta \\ 1 & -\beta & -\beta & -\beta & -\beta & \beta^2 \\ -\alpha & -\alpha & \alpha^2 & 1 & -\alpha & -\alpha \\ \alpha^2 & -\alpha & -\alpha & -\alpha & -\alpha & 1 \end{pmatrix} \begin{pmatrix} p_{01}^2 \\ p_{02}^2 \\ p_{03}^2 \\ p_{12}^2 \\ p_{13}^2 \\ p_{23}^2 \end{pmatrix} = 0\,.$$

Permute the variables into the order $(p_{02}, p_{13}, p_{03}, p_{12}, p_{01}, p_{23})$. For α, β satisfying

$$\alpha\beta(1-\alpha\beta)(1+\beta)(1+\alpha) \neq 0\,, \tag{5}$$

Gaussian elimination yields the following system:

$$\begin{pmatrix} -\beta & -\beta & (1-\alpha)(1-\beta) & 0 & 0 & 0 \\ 0 & 0 & \alpha & -\beta & 0 & 0 \\ 0 & 0 & 0 & -\beta & \alpha & 0 \\ 0 & 0 & 0 & 0 & \alpha & -\beta \end{pmatrix} \begin{pmatrix} p_{02}^2 \\ p_{13}^2 \\ p_{03}^2 \\ p_{12}^2 \\ p_{01}^2 \\ p_{23}^2 \end{pmatrix} = 0\,.$$

Together with the Plücker equation (1), this gives the following system of equations:

$$-\beta p_{02}^2 - \beta p_{13}^2 + (1-\alpha)(1-\beta)p_{03}^2 \;=\; 0\,, \tag{6}$$
$$p_{01}p_{23} - p_{02}p_{13} + p_{03}p_{12} \;=\; 0\,, \tag{7}$$
$$\alpha p_{01}^2 \;=\; \alpha p_{03}^2 \;=\; \beta p_{12}^2 \;=\; \beta p_{23}^2\,. \tag{8}$$

For α, β satisfying (5) as well as $(1-\alpha)(1-\beta) \neq 0$, we distinguish the following three disjoint cases.

Case 1: $p_{02} = 0$.
Since $p_{13} = 0$ would imply that all components are zero and hence contradict $(p_{01}, \ldots, p_{23})^{\mathrm{T}} \in \mathbb{P}^5$, we can assume $p_{13} = 1$. Then (6) and (8) imply

$$\alpha p_{01}^2 \;=\; \alpha p_{03}^2 \;=\; \beta p_{12}^2 \;=\; \beta p_{23}^2 \;=\; \frac{\alpha\beta}{(1-\alpha)(1-\beta)} \neq 0\,.$$

Since (7) implies $\mathrm{sgn}(p_{01}p_{23}) = -\mathrm{sgn}(p_{03}p_{12})$, only 8 of the $2^4 = 16$ sign combinations for $p_{01}, p_{03}, p_{12}, p_{23}$ are possible. Namely, the 8 (complex) solutions for $p_{01}, p_{03}, p_{12}, p_{23}$ are

$$(p_{01}, p_{03}, p_{12}, p_{23})^{\mathrm{T}} = \frac{1}{\sqrt{(1-\alpha)(1-\beta)}}\,(\gamma_{01}\beta, \gamma_{03}\beta, \gamma_{12}\alpha, -\mathrm{sgn}(\gamma_{01}\gamma_{03}\gamma_{12})\alpha)^{\mathrm{T}} \tag{9}$$

with $\gamma_{01}, \gamma_{03}, \gamma_{12} \in \{-1, 1\}$. Hence, for $\alpha, \beta \in \mathbb{R}^2$ satisfying (5), this case gives 8 distinct common tangents.

Case 2: $p_{13} = 0$.
This case is symmetric to case 1. Setting $p_{13} = 1$, the resulting 8 solutions for the variables $p_{01}, p_{03}, p_{12}, p_{23}$ are the same ones as in (9).

Case 3: $p_{02}p_{13} \neq 0$.
Without loss of generality, we can assume $p_{02} = 1$. Solving (7) for p_{13} and substituting this expression into (6) yields

$$-\beta - \beta p_{01}^2 p_{23}^2 - \beta p_{03}^2 p_{12}^2 - 2\beta p_{01}p_{03}p_{12}p_{23} + (1-\alpha)(1-\beta)p_{03}^2 \;=\; 0\,.$$

We use (8) to write this in terms of p_{01}. This is straightforward for the squared terms, but for the other terms, we observe that, by (8), $p_{01}p_{23} = \pm p_{03}p_{12}$ and since $p_{02}p_{13} \neq 0$, the Plücker equation (7) implies these have the same sign. This gives the quartic equation in p_{01}

$$-\beta + (1-\alpha)(1-\beta)p_{01}^2 - 4\alpha p_{01}^4 \;=\; 0\,,$$

whose discriminant is

$$(1-\alpha)^2(1-\beta)^2 - 16\alpha\beta\,. \tag{10}$$

Hence, for $\alpha, \beta \in \mathbb{R}^2$ satisfying (5), and for which this discriminant does not vanish, there are two different solutions for p_{01}^2. For each of these two solutions for p_{01}^2, there are 8 distinct solutions for $p_{01}, p_{03}, p_{12}, p_{23}$, namely

$$(p_{01}, p_{03}, p_{12}, p_{23})^{\mathrm{T}} \;=\; \sqrt{p_{01}^2}\,(\gamma_{01}, \gamma_{03}, \gamma_{12}, \mathrm{sgn}(\gamma_{01}\gamma_{03}\gamma_{12}))^{\mathrm{T}} \tag{11}$$

with $\gamma_{01}, \gamma_{03}, \gamma_{12} \in \{-1, 1\}$. Since p_{13} is uniquely determined by p_{01}, p_{02}, p_{03}, p_{12}, case 3 gives 16 distinct common tangents.

In order to determine when all solutions are real, suppose first that $\alpha = \beta$. Then the discriminant (10) becomes $(\alpha^2 - 6\alpha + 1)(\alpha + 1)^2$, and its smallest positive root is $\alpha_0 := 3 - 2\sqrt{2} \approx 0.17157$. In particular, for $0 < \alpha < \alpha_0$, the discriminant in case 3 is positive and both solutions for p_{01}^2 are positive. Thus, for $0 < \beta = \alpha < \alpha_0$, the solutions of all three cases are distinct and real. Next, fix $0 < \alpha < \alpha_0$ and suppose that $0 < \beta < \alpha$. Then the discriminant (10) is positive: for fixed $0 < \alpha < \alpha_0$, the discriminant (10) is decreasing in β for $0 < \beta < \alpha$ and positive when $\beta = \alpha$. This concludes the proof of Theorem 8. $\square$

Figure 2 illustrates the construction and the 32 tangents for $\alpha = 1/10$ and $\beta = 1/20$.

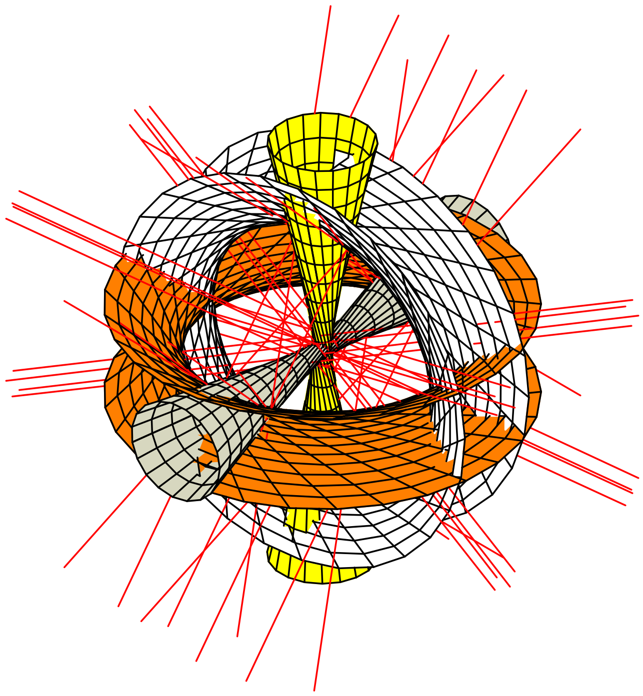

FIGURE 2. The configuration of quadrics from Theorem 8.

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