RICCI CURVATURE RIGIDITY FOR WEAKLY ASYMPTOTICALLY HYPERBOLIC MANIFOLDS

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Abstract. We obtain rigidity results for Riemannian manifolds which are weakly asymptotically hyperbolic and have lower bound on Ricci curvature. Our argument consists of two steps. First we compactify the metric by its positive eigenfunction. Then we apply a quasi-local mass characterization of Euclidean balls to the compactified metric. As a result, a weak asymptotic condition on the metric is obtained to assure the rigidity.

1. Introduction

Rigidity questions for asymptotically hyperbolic manifolds have been studied by many authors under various assumptions. In [9], Min-Oo proved a scalar curvature rigidity theorem for manifolds which are spin and asymptotic to the hyperbolic space in a strong sense. In [2], Andersson and Dahl improved Min-Oo's result to asymptotically locally hyperbolic spin manifolds. They also obtained rigidity results for conformally compact Einstein manifolds with spin structure. More recent related works under spin assumption can be found in [4], [15] and [16]. It is interesting to know if similar results still hold without spin assumption. In [7], Listing proved a non-spin rigidity theorem for manifolds with lower bound on sectional curvature. In [10], Qing established a rigidity theorem for conformally compact Einstein manifolds of dimension less than 8. Under the same dimension assumption, Andersson, Cai and Galloway [1] recently have proved rigidity results for manifolds which are exactly the hyperbolic space outside a compact set and have lower bound on scalar curvature. For general dimension, Shi and Tian [14] obtained a Ricci curvature rigidity theorem for manifolds whose exponential maps are diffeomorphism and satisfy an asymptotic sectional curvature decay condition along outgoing geodesics.

Our research in this paper is inspired by the method of [10]. There are two important ingredients in [10]. One is conformally compactifying a metric by its positive eigenfunction. The other one is the classic positive mass theorem proved by Schoen and Yau [12] for asymptotically flat manifolds. By exploiting a quasi-local mass characterization of Euclidean balls, which is essentially a localized version of the Positive Mass Theorem [8] [13], we obtain a Ricci curvature rigidity result for manifolds which satisfy an asymptotic Ricci curvature decay condition.

Before we state our theorems, we first explain what *asymptotically hyperbolic* means in our setting. There are different ways to formulate this concept and we chose to work with spaces which are *conformally compact*. Let $\bar{X}^{n+1}$ be a compact manifold

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with boundary ∂X and interior X^{n+1}. A smooth Riemannian metric g, defined on X, is said to be *conformally compact of order* $C^{m,\alpha}$ if $\bar{g} = \rho^2 g$ extends to a $C^{m,\alpha}$ metric on $\bar{X}$ for any smooth defining function ρ of ∂X, in the sense that $\rho > 0$ on X, $\rho = 0$ and $d\rho \neq 0$ at ∂X. The metric $\bar{g}$ restricted to ∂X induces a metric $\hat{g}$ on ∂X which rescales upon change in defining function. Therefore a conformally compact metric g defines a conformal structure on ∂X. We call $(\partial X, [\hat{g}])$ the *conformal infinity of* (X, g). If $m + \alpha \geq 2$, straightforward computation as in (28) shows that $K(x)$, the sectional curvature of g at $x \in X$, approaches $-|d\rho|^2_{\bar{g}}$ at ∂X as x approaches ∂X. Accordingly, we have the following definition.

Definition 1. *A Riemannian manifold* (X^{n+1}, g) *is said to be* weakly asymptotically hyperbolic of order $C^{m,\alpha}$ *if* g *is conformally compact of order* $C^{m,\alpha}$ *for some* $m+\alpha \geq 2$ *and* $|d\rho|^2_{\bar{g}} = 1$ *at* ∂X.

Remark 1. *The condition* $|d\rho|^2_{\bar{g}} = 1$ *at* ∂X *alone only implies that* $|K(x) + 1|$ *grows at most like* ρ^2*, as shown in (27).*

Next we introduce function spaces compatible with the asymtotic behavior of g as ρ tends to zero. A function $u \in C^{m,\alpha}(X)$ is said to be in the *weighted Hölder space* $\Lambda^\delta_{m,\alpha}(X)$ if $||u||^\delta_{m,\alpha} < \infty$ for $\delta \in R, m \geq 0$ and $\alpha \in (0, 1)$, where the norm $||u||^\delta_{m,\alpha}$ is defined as follows. First, in the special case in which X is a smoothly bounded open subset of $\mathbb{R}^{n+1}$, we define

$$||u||^\delta_{m,0} = \sum_{l=0}^{m} \sum_{|\gamma|=l} ||d^{-\delta+l} \partial^\gamma u||_{L^\infty}$$

and

$$||u||^\delta_{m,\alpha} = ||u||^\delta_{m,0} + \sum_{|\gamma|=m} \sup_{x,y} \left[\min(d_x^{-\delta+m+\alpha}, d_y^{-\delta+m+\alpha}) \frac{|\partial^\gamma u(x) - \partial^\gamma u(y)|}{|x-y|^\alpha} \right],$$

where d_x is the Euclidean distance from x to ∂X. In the more general cases of a manifold with boundary, the same norms are defined using a covering by coordinate charts and a subordinate partition of unity argument. We recommend [5] and [6] for succinct discussions of properties of the spaces $\Lambda^\delta_{m,\alpha}(X)$.

Throughout this paper, we let $S^n \subset \mathbb{R}^{n+1}$ be the unit sphere and h_0 be the induced metric. For notation convenience and possible later generalization, we call an integer n *a PMT integer* if the the Positive Mass Theorem [12] holds for asymptotically flat manifolds whose dimension are n. (For instance, any $3 \leq n \leq 7$ is a PMT integer.)

Now we are in a position to state our theorem.

Theorem 1. *Let* (X^{n+1}, g) *be a conformally compact manifold of order* $C^{3,\alpha}$ *which satisfies* $Ric(g) \geq -ng$*. Suppose that* (X^{n+1}, g) *has the standard round sphere* $(S^n, [h_0])$ *as its conformal infinity and*

$$|Ric + n| \in \Lambda^\delta_{0,\beta}(X) \tag{1}$$

for some $0 < \beta < 1$, $\delta > 2$, *where* $|Ric+n|(x) = \sup_{v \in T_x X, ||v||=1}\{|Ric(v,v)+n|\}$, *then* (X^{n+1}, g) *is isometric to the hyperbolic space* $\mathbb{H}^{n+1}$ *provided* $n+1$ *is a PMT integer.*

We will prove Theorem 1 by establishing a stronger result which requires weaker asymptotic condition on the metric.

Theorem 2. *Let (X^{n+1}, g) be a weakly asymptotically hyperbolic manifold of order $C^{3,\alpha}$ which satisfies $Ric(g) \geq -ng$. Assume that (X^{n+1}, g) has the standard round sphere $(S^n, [h_0])$ as its conformal infinity. Let r be a special defining function of ∂X such that*

$$g = \frac{1}{\sinh^2(r)}\{dr^2 + g_r\} \tag{2}$$

in a neighborhood of ∂X and $g_0 = h_0$. Then, if

$$Tr_{g_r}(\frac{d}{dr}g_r) \in \Lambda^{\delta}_{0,\beta}(X) \tag{3}$$

for some $0 < \beta < 1$, $\delta > 1$, (X^{n+1}, g) is isometric to the hyperbolic space $\mathbb{H}^{n+1}$ provided $n+1$ is a PMT integer.

The rest of the paper is organized as follows. In Section 2, we recall some analytic and geometric preliminaries. In Section 3, we perform the conformal compactification and prove Theorem 2. In Section 4, we recall some basic calculation in conformal geometry and derive Theorem 1. We conclude the paper in Section 5 by comparing our research to that of establishing a Positive Mass Theorem on asymptotically hyperbolic manifolds in [4], [15] and [16].

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2. Analytic and Geometric Preliminaries

We first recall the following lemma from, for instance, [5] [6], etc.

Lemma 1. *Let (X, g) be a weakly asymptotically hyperbolic manifold of order $C^{3,\alpha}$. Then any representative $\hat{g}$ of the conformal infinity of g determines a unique defining function $s \in C^{2,\alpha}(\bar{X})$ such that $s^2 g|_{\partial X} = \hat{g}$, $s^2 g$ has a $C^{2,\alpha}$ extension to $\bar{X}$ and $|ds|^2_{s^2 g} \equiv 1$ on a neighborhood U of ∂X in $\bar{X}$. Hence, s gives an identification of U with $\partial X \times [0, \epsilon)$, for some $\epsilon > 0$, such that*

$$g = \frac{1}{s^2}(ds^2 + g_s) \tag{4}$$

for a 1-parameter family $\{g_s\}$ of metrics on ∂X with $g_0 = \hat{g}$.

By a change of variable

$$s = \frac{\cosh(r) - 1}{\sinh(r)}, \tag{5}$$

we can rewrite (4) as

$$g = \rho^{-2}(dr^2 + g_r), \tag{6}$$

where $\rho = \sinh(r)$. One may compare (6) with the fact that

$$g_b = \frac{1}{\sinh^2(r)}\{dr^2 + h_0\}$$

gives the standard hyperbolic metric on $S^n \times \mathbb{R}^+$. The fact that s is $C^{2,\alpha}$ guarantees that the family of metrics $\{g_r\}$ is at least C^1 with respect to r. In the special case in which (X^{n+1}, g) is Einstein and conformally compact of sufficiently high order, Andersson and Dahl [2] showed that the family of metrics $\{g_r\}$ in (6) have the properties

$$g_r = h_0 + \rho^n h, \ Tr_{h_0} h = O(\rho^n), \ \rho = \sinh(r). \tag{7}$$

Thus, the decay assumption (3) is automatically satisfied by any conformally compact Einstein manifold with the round sphere as its conformal infinity. Next we recall an analytic result of the operator $-\triangle_g + (n+1)$ between suitable weighted Hölder spaces (Proposition 3.3 in [6]).

Lemma 2. *Let (X^{n+1}, g) be weakly asymptotically hyperbolic of order $C^{m,\alpha}$. Let $0 < \beta < 1$ and $k + 1 + \beta \leq m + \alpha$. Then*

$$-\triangle + (n+1) : \Lambda^\delta_{k+2,\beta} \to \Lambda^\delta_{k,\beta}$$

is an isomorphism whenever $-1 < \delta < n+1$.

The Positive Mass Theorem [12] is applied in [10] to the doubling of a partially compactified manifold along its totally geodesic boundary. Here we observe that it would be much simpler if we appeal to the following quasi-local mass type result proved in [8] [13](see also [11]).

Proposition 1. *Let $\bar{\Omega}^{n+1}$ be a smooth compact manifold with boundary $\partial\Omega$. Let g be a metric on $\bar{\Omega}$ which is smooth in the interior Ω and C^2 up to $\partial\Omega$. If g has nonnegative scalar curvature in Ω, $(\partial\Omega, g|_{T\partial\Omega})$ is isometric to (S^n, h_0) and the mean curvature of $\partial\Omega$ with respect to the outward pointing unit normal identically equals the constant n, then g has vanishing scalar curvature in Ω provided $n+1$ is a PMT integer.*

Remark 2. *It is desirable to further conclude that g is actually flat on Ω which is indeed the case when $n = 2$ [8]. However, no proof in higher dimension is known so far except the case when (Ω, g) is assumed to be spin [13].*

We conclude this section by recalling a nice functional characterization of the Hyperbolic space $\mathbb{H}^{n+1}$ proved in [11].

Lemma 3. *Let (X^{n+1}, g) be a complete Riemannian manifold. Assume that there exits a positive smooth function u on X such that*

$$Hess_g(u) = ug.$$

Then (X^{n+1}, g) is isometric to $(\mathbb{H}^{n+1}, g_{\mathbb{H}})$.

3. Ball Type Compactification

Let (X^{n+1}, g) satisfy the assumptions in Theorem 2 and let U be a neighborhood of ∂X in $\bar{X}$ where (2) holds. We introduce a background hyperbolic metric

$$g_b = \frac{1}{\rho^2}\{dr^2 + h_0\} \tag{8}$$

on U. Clearly (U, g_b) can be identified with the complement of some compact set in the Hyperbolic space $(\mathbb{H}^{n+1}, g_{\mathbb{H}})$ realized as the hypersurface

$$\{(x_1, \ldots, x_{n+1}, t) \mid |x|^2 - t^2 = -1, t > 0\} \subset \mathbb{R}^{n+1,1}$$

by letting $\sinh r = \rho = \frac{1}{|x|}$. The restriction of t to $\mathbb{H}^{n+1}$ is an eigenfunction of $(\mathbb{H}^{n+1}, g_{\mathbb{H}})$, i.e.

$$\Delta_{\mathbb{H}} t = (n+1)t.$$

Moreover, by a change of variables

$$t = \frac{1 + |y|^2}{1 - |y|^2},$$

$(\mathbb{H}^{n+1}, g_{\mathbb{H}})$ can be rewritten as $(B^{n+1}, (\frac{2}{1-|y|^2})^2 |dy|^2)$, which implies that $g_{\mathbb{H}}$ can be compactified to be the the standard Euclidean metric by $\frac{1}{(1+t)^2}$. This leads us to transplant t to the domain U and then look for a positive eigenfunction u on (X^{n+1}, g) which behaves like t near ∂X. To simplify notations, we use $v \in O_k(\rho^\delta)$ to standard for $v \in \Lambda^\delta_{k,\beta}(X)$ for $k \geq 0$ and a fixed $0 < \beta < 1$.

Lemma 4. *There exists a smooth function $u > 0$ on (X, g) such that*

$$-\triangle_g u + (n+1)u = 0 \tag{9}$$

and

$$u = t + O_2(\rho^{\tilde{\delta}}) \tag{10}$$

for some $1 < \tilde{\delta} < n + 1$.

Proof: The fact $r \in C^2(U)$ and $t = \sqrt{1 + \frac{1}{\rho^2}}$ implies $t \in C^2(U)$. We calculate

$$\begin{aligned} -\triangle_g t &= -\frac{\rho^{n+1}}{\sqrt{detg_r}} \partial_r (\rho^{1-n} \sqrt{detg_r} \partial_r t) \\ &= -(n+1)t + \frac{1}{2} Tr_{g_r} g_r', \end{aligned} \tag{11}$$

where "$\prime$" denotes differentiation with respect to r. By the decay assumption (3), we may choose $1 < \tilde{\delta} < \min(\delta, n+1)$ such that $Tr_{g_r} g_r' = O_0(\rho^{\tilde{\delta}})$. Hence, by Lemma 2, we know there exists a function $w = O_2(\rho^{\tilde{\delta}})$ such that

$$-\triangle_g w + (n+1)w = -\triangle_g t + (n+1)t. \tag{12}$$

Let $u = t - w$. Then $u > 0$ by the maximum principle and the smoothness of u follows directly from the local elliptic regularity theory. □

We refer readers to [6], [10] and [3] for more results on eigenfunctions for asymptotically hyperbolic manifolds. In our next lemma, we set the stage to apply the work of [8].

Lemma 5. *The metric $g_u = \frac{1}{(1+u)^2} g$ extends to a C^2 metric on $\bar{X}$ such that g_u has nonnegative scalar curvature in X, $(\partial X, g_u|_{\partial X})$ is isometric to (S^n, h_0) and the mean curvature of ∂X in $(\bar{X}, g_u)$ identically equals the constant n.*

Proof: First we calculate the scalar curvature of g_u,

$$\begin{aligned} R(g_u) &= \frac{4n}{1-n}(u+1)^{\frac{n+3}{2}}[\triangle_g - \frac{n-1}{4n}R(g)](u+1)^{\frac{1-n}{2}} \\ &= -n(n+1)|du|_g^2 + 2n(n+1)u(u+1) + R(g)(u+1)^2. \end{aligned} \tag{13}$$

Since $Ric(g) \geq -ng$, we have $R(g) \geq -n(n+1)$ so (13) implies

$$\begin{aligned} R(g_u) &\geq -n(n+1)|du|_g^2 + 2n(n+1)u(u+1) - n(n+1)(u+1)^2 \\ &= n(n+1)(u^2 - |du|_g^2 - 1). \end{aligned} \tag{14}$$

As in [10], we then appeal to the Bochner formula for eigenfunctions, which is observed in [6].

$$\begin{aligned} \triangle_g(|du|_g^2 - u^2) &= 2n|du|_g^2 + 2Ric(\nabla_g u, \nabla_g u) + 2|Hess_g u|_g^2 - 2(n+1)u^2 \\ &\geq 2|Hess_g u|_g^2 - 2(n+1)u^2, \end{aligned} \tag{15}$$

where the last step holds again since $Ric(g) \geq -ng$. Therefore, we have

$$-\triangle_g(u^2 - |du|_g^2 - 1) \geq 2|Hess_g u - ug|_g^2. \tag{16}$$

Hence, in order to prove the scalar curvature $R(g_u) \geq 0$, we only need to apply a maximum principle to $u^2 - |du|^2 - 1$ and verify that it goes to zero towards the boundary. A straightforward calculation reveals that

$$\begin{aligned} u^2 - |du|_g^2 &= t^2 - 2tw + w^2 - \rho^2(\partial_r t)^2 - \rho^2 2\partial_r t \partial_r w \\ &\quad -\rho^2(\partial_r w)^2 - g^{\delta\lambda}\partial_\delta w \partial_\lambda w \\ &= 1 + O_2(\rho^{\tilde{\delta}-1}) + O_2(\rho^{2\tilde{\delta}}) + O_1(\rho^{\tilde{\delta}-1}) \\ &\quad +O_1(\rho^{2\tilde{\delta}}) + O_1(\rho^{2\tilde{\delta}}), \end{aligned} \tag{17}$$

where we have used the fact $t^2 - \rho^2(\partial_r t)^2 = 1$. It follows from $\tilde{\delta} > 1$ that

$$u^2 - |du|_g^2 - 1 \to 0, \quad as\ \rho \to 0. \tag{18}$$

Thus, we have

$$u^2 - |du|_g^2 - 1 \geq 0 \quad on\ X, \tag{19}$$

which implies $R(g_u) \geq 0$ on X by (14).

Next we consider the expansion of g_u near ∂X,

$$g_u = \frac{1}{[(u+1)\rho]^2}\{dr^2 + g_r\}, \tag{20}$$

where

$$(u+1)\rho = \cosh r + \sinh r - w \sinh r. \tag{21}$$

Since $w = O_2(\rho^{\tilde{\delta}}), \tilde{\delta} > 1$ and $\rho^2 g = dr^2 + g_r$ is C^2 on $\bar{X}$, we see that g_u readily extends to a C^2 metric on $\bar{X}$. Furthermore, we have the boundary values

$$(u+1)\rho|_{r=0} = 1 \quad and \quad \frac{d}{dr}[(u+1)\rho]|_{r=0} = 1, \tag{22}$$

which, combined with the facts $g_0 = h_0$ and $Tr_{g_0} g_0' = 0$, show that $(\partial X, g_u|TM)$ is isometric to (S^n, h_0) and ∂X has constant mean curvature n in $(\bar{X}, g_u)$.

Now it follows from Proposition 1 that $R(g_u) \equiv 0$ on X. (14), (19) and (16) then imply that

$$|Hess_g u - ug| = 0. \tag{23}$$

Therefore (X^{n+1}, g) is the hyperbolic space $\mathbb{H}^{n+1}$ by Lemma 3. □

Remark 3. *We can reformulate the decay assumption (3) in terms of the metric expansion (4). By substituting $s = \frac{\cosh(r)-1}{\sinh(r)}$ back we see that (3) is equivalent to*

$$Tr_{g_s}(\frac{d}{ds} g_s) + ns \in \Lambda^{\delta}_{0,\beta}(X) \tag{24}$$

for some $\delta > 1$.

4. Asymptotic Condition in terms of Curvature

We prove Theorem 1 in this section . Our calculation shows that a decay condition of Ricci curvature is enough to assure rigidity.

Given any defining function ρ, let $\bar{g} = \rho^2 g$ and $\{\omega_a\}_{0 \le a \le n}$ be a local orthonormal coframe of $\bar{g}$. Define $\eta_a = \rho^{-1}\omega_a$, then $\{\eta_a\}_{0 \le a \le n}$ forms a local orthonormal coframe of g. Let $\bar{R}_{abcd}$, R_{abcd} denote the component of the curvature tensor of $\bar{g}$, g in the coframe $\{\omega_a\}_{0 \le a \le n}$, $\{\eta_a\}_{0 \le a \le n}$. Straightforward calculation shows that

$$\begin{aligned} R_{abcd} \;&=\; \rho^2 \bar{R}_{abcd} + \rho^2 \sum_{m=0}^{m=n} [(\log\rho)_{;bm} + (\log\rho)_{;m}(\log\rho)_{;b}](\delta_{ac}\delta_{dm} - \delta_{ad}\delta_{mc}) \\ &\quad + \rho^2 \sum_{m=0}^{m=n} [(\log\rho)_{;am} + (\log\rho)_{;m}(\log\rho)_{;a}](\delta_{mc}\delta_{db} - \delta_{md}\delta_{bc}) \\ &\quad - \rho^2 |\nabla_{\bar{g}} \log\rho|^2 (\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}), \end{aligned} \tag{25}$$

where ";" denotes covariant differentiation with respect to $\bar{g}$. It follows that

$$\begin{aligned} R_{abab} \;&=\; \rho^2 \bar{R}_{abab} + \rho^2[(\log\rho)_{;bb} + (\log\rho)^2_{;b}] \\ &\quad + \rho^2[(\log\rho)_{;aa} + (\log\rho)^2_{;a}] - \rho^2|\nabla_{\bar{g}} \log\rho|^2 \\ &=\; \rho^2 \bar{R}_{abab} + \rho(\rho_{;aa}) + \rho(\rho_{;bb}) - \rho^2|\nabla_{\bar{g}} \log\rho|^2 \end{aligned} \tag{26}$$

where $a \neq b$. Hence,

$$\frac{R_{abab} + |\nabla_{\bar{g}}\rho|^2}{\rho^2} = \bar{R}_{abab} + \frac{(\rho_{;aa} + \rho_{;bb})}{\rho}, \tag{27}$$

which implies that

$$R_{abab} + |\nabla_{\bar{g}}\rho|^2 = O(\rho), \tag{28}$$

provided $\bar{g} \in C^2(\bar{X})$. Taking trace of (27), we obtain that

$$\frac{Ric_{aa} + n|\nabla_{\bar{g}}\rho|^2}{\rho^2} = \overline{Ric}_{aa} + \frac{[(n-1)\rho_{;aa} + \sum_{b=1}^{n+1} \rho_{;bb}]}{\rho}, \tag{29}$$

where $Ric_{aa}, \overline{Ric}_{aa}$ denotes the Ricci curvature tensor of $g, \bar{g}$. We can rewrite (29) as

$$\frac{Ric_{aa}+n}{\rho^2} = \overline{Ric}_{aa} + \frac{[(n-1)\rho_{;aa} + \triangle_{\bar{g}}\rho]}{\rho} + \frac{n(1-|\nabla_{\bar{g}}\rho|^2)}{\rho^2}. \tag{30}$$

Taking trace of (30), we obtain that

$$\frac{R+n(n+1)}{\rho^2} = \bar{R} + 2n\frac{\triangle_{\bar{g}}\rho}{\rho} + n(n+1)\frac{(1-|\nabla_{\bar{g}}\rho|^2)}{\rho^2}, \tag{31}$$

where $R, \bar{R}$ denotes the scalar curvature of $g, \bar{g}$. The following lemma follows directly from (31).

Lemma 6. *If (X^{n+1}, g) is conformally compact of order $C^{2,\alpha}$ and*

$$R + n(n+1) = O(\rho),$$

then $|\nabla_{\bar{g}}\rho|^2 = 1$ on ∂X, i.e (X, g) is weakly asymptotically hyperbolic.

Next we assume that (X^{n+1}, g) is conformally compact of order $C^{3,\alpha}$ and

$$|Ric + n| = O(\rho^\delta) \tag{32}$$

for some $\delta > 2$. It follows from Lemma 6 and Lemma 1 that there exists s, a defining function of ∂X, such that $\bar{g} = s^2 g \in C^{2,\alpha}(\bar{X})$ and

$$\bar{g} = ds^2 + g_s,$$

where g_s is the induced metric on the level set Σ_s of s near ∂X. For this fixed $\bar{g}$, we choose $\omega_0 = ds$ and require ω_i to be tangent to Σ_s, where $i \in \{1, 2, \ldots, n\}$. Let $A_{ij}(s)$, $H(s)$ denote the second fundamental form, the mean curvature of Σ_s in $(\bar{X}, \bar{g})$, then

$$s_{;ij} = A_{;ij}, \quad Tr_{g_s}\{\frac{d}{ds}g_s\} = 2H(s) = 2\sum_{i=1}^{i=n} A_{ii} = 2\triangle_{\bar{g}}s. \tag{33}$$

By Remark 2, we are interested in $2H(s) + ns$. We note that (30) and (31) are reduced to

$$\frac{Ric_{aa}+n}{s^2} = \overline{Ric}_{aa} + \frac{[(n-1)s_{;aa} + H(s)]}{s} \tag{34}$$

and

$$\frac{R+n(n+1)}{s^2} = \bar{R} + 2n\frac{H(s)}{s}, \tag{35}$$

which implies that $H(s) = O(s)$ and $|A(s)|^2 = \sum_{i=1}^{i=n} s_{;ii}^2 = O(s)$. On the other hand, it follows from Gauss equation that

$$\bar{R}_s = \bar{R} - 2\overline{Ric}_{00} + H(s)^2 - |A(s)|^2, \tag{36}$$

where $\bar{R}_s$ is the scalar curvature of g_s and Ric_{00} is given by (34),

$$\frac{Ric_{00}+n}{s^2} = \overline{Ric}_{00} + \frac{H(s)}{s}. \tag{37}$$

Therefore, it follows from (35), (36) and (37) that

$$(n-1)\frac{(2H(s)+ns)}{s} = -\frac{2(Ric_{00}+n)}{s^2}+\frac{[R+n(n+1)]}{s^2} + H(s)^2-|A(s)|^2+[n(n-1)-\bar{R}_s]. \tag{38}$$

Hence, we conclude that

Lemma 7. *Suppose* $|Ric+n| = O(r^{2+\delta})$ *for some* $\delta > 0$. *Let* s *be a defining function of* $\bar{X}$ *such that* $g = \frac{1}{s^2}\{ds^2+g_s\}$ *and* g_0 *has constant scalar curvature* $n(n-1)$. *Then* $Tr_{g_s}\{\frac{d}{ds}g_s + ns\} = O(r^{1+\epsilon})$ *for some* $\epsilon > 0$.

Theorem 1 now follows readily from Remark 3, Lemma 7 and Theorem 2. □

5. Remarks

To conclude we want to make some remarks. The results in this paper improve the rigidity theorem in [10]. In particular we obtain weaker asymptotic condition at the infinity to assure the rigidity, which can be interpretated as assuming the Einstein equations are satisfied at the infinity to a very low order. This is of interest since the energy-momentum tensor usually vanishes to certain order for isolated systems. Also, we believe it is interesting to compare our result to those on the Positive Mass Theorem for asymptotically hyperbolic manifolds in [4], [15], [16]. In [15], Wang defines a conformally compact manifold (X^{n+1}, g) to be *asymptotically hyperbolic* if

(1) (X^{n+1}, g) is weakly asymptotically hyperbolic with the conformal infinity being the standard sphere (S^n, h_0).
(2) Let r be the special defining function so that we can write

$$g = \frac{1}{\sinh^2(r)}\{dr^2 + g_r\}$$

in a neighborhood of ∂X. Then

$$g_r = h_0 + \frac{r^{n+1}}{n+1}h + O(r^{n+2}),$$

where h is a symmetric 2-tensor on S^n. Moreover the asymptotic expansion can be differentiated twice.

Working with this definition, see also [2], [4] and [9], Wang was able to prove that if (X^{n+1}, g) is asymptotically hyperbolic, (X^{n+1}, g) is spin and the scalar curvature $R \geq -n(n+1)$, then

$$\int_{S^n}(Tr_{h_0}h)d\mu_{h_0} \geq \left|\int_{S^n}(Tr_{h_0}h)x\,d\mu_{h_0}\right|.$$

Moreover equality holds if and only if (X, g) is isometric to the hyperbolic space $\mathbb{H}^{n+1}$. Since our decay assumption in Theorem 2 is much weaker than (2), we immediately have the following corollary,

Corollary. *Assume that* (X^{n+1}, g) *is an asymptotically hyperbolic manifold* [15] *such that* $Ric \geq -ng$, *then* (X^{n+1}, g) *is isometric to the hyperbolic space* $\mathbb{H}^{n+1}$ *provided* $n+1$ *is a PMT integer.*

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