# ON THE HOMOTOPY LIE ALGEBRA OF AN ARRANGEMENT

GRAHAM DENHAM[1] AND ALEXANDER I. SUCIU[2]

Abstract. Let $A$ be a graded-commutative, connected $\Bbbk$-algebra generated in degree 1. The homotopy Lie algebra $\mathfrak{g}_A$ is defined to be the Lie algebra of primitives of the Yoneda algebra, $\operatorname{Ext}_A(\Bbbk, \Bbbk)$. Under certain homological assumptions on $A$ and its quadratic closure, we express $\mathfrak{g}_A$ as a semi-direct product of the well-understood holonomy Lie algebra $\mathfrak{h}_A$ with a certain $\mathfrak{h}_A$-module. This allows us to compute the homotopy Lie algebra associated to the cohomology ring of the complement of a complex hyperplane arrangement, provided some combinatorial assumptions are satisfied. As an application, we give examples of hyperplane arrangements whose complements have the same Poincaré polynomial, the same fundamental group, and the same holonomy Lie algebra, yet different homotopy Lie algebras.

## 1. Definitions and statements of results

1.1. **Holonomy and homotopy Lie algebras.** Let $A$ be a graded, graded-commutative algebra over a field $\Bbbk$, with graded piece $A_k$, $k \geq 0$. We will assume that $A$ is locally finite, connected, and generated in degree 1. In other words, $A = T(V)/I$, where $V$ is a finite-dimensional $\Bbbk$-vector space, $T(V) = \bigoplus_{k\geq 0} V^{\otimes k}$ is the tensor algebra on $V$, and $I$ is a two-sided ideal, generated in degrees 2 and higher. To such an algebra $A$, one naturally associates two graded Lie algebras over $\Bbbk$ (see for instance [12]).

**Definition 1.1.** The *holonomy Lie algebra* $\mathfrak{h}_A$ is the quotient of the free Lie algebra on the dual of $A_1$, modulo the ideal generated by the image of the transpose of the multiplication map $\mu\colon A_1 \wedge A_1 \to A_2$:

$$\mathfrak{h}_A = \operatorname{Lie}(A_1^*)\big/\operatorname{ideal}\left(\operatorname{im}(\mu^*\colon A_2^* \to A_1^* \wedge A_1^*)\right). \tag{1}$$

Note that $\mathfrak{h}_A$ depends only on the quadratic closure of $A$: if we put $\overline{A} = T(V)/(I_2)$, then $\mathfrak{h}_A = \mathfrak{h}_{\overline{A}}$.

**Definition 1.2.** The *homotopy Lie algebra* $\mathfrak{g}_A$ is the graded Lie algebra of primitive elements in the Yoneda algebra of $A$:

$$\mathfrak{g}_A = \operatorname{Prim}(\operatorname{Ext}_A(\Bbbk, \Bbbk)). \tag{2}$$

---

2000 *Mathematics Subject Classification.* Primary 16E05, 52C35; Secondary 16S37, 55P62.

*Key words and phrases.* Holonomy and homotopy Lie algebras, hyperplane arrangement, supersolvable, hypersolvable, Yoneda algebra, Koszul algebra, Hopf algebra, spectral sequence, homotopy groups.

[1]Partially supported by a grant from NSERC of Canada.

[2]Partially supported by NSF grant DMS-0311142.

In other words, the universal enveloping algebra of the homotopy Lie algebra is the Yoneda algebra:

$$U(\mathfrak{g}_A) = \mathrm{Ext}_A(\Bbbk, \Bbbk). \tag{3}$$

The algebra $U = \mathrm{Ext}_A(\Bbbk, \Bbbk)$ is a bigraded algebra; let us write $U^{pq}$ to denote cohomological degree $p$ and polynomial degree $q$. Then $U^{pq} = 0$, unless $-q \geq p$. The subalgebra $R = \bigoplus_{p\geq 0} U^{p,-p}$ is called the linear strand of $U$. For convenience, we will let $U^p_q = U^{p,-p-q}$. The lower index $q$ is called the internal degree. Then $U$ is a graded $R$-algebra, with $R = U_0$.

The relationship between the holonomy and homotopy Lie algebras of $A$ is provided by the following well-known result of Löfwall.

**Lemma 1.3** (Löfwall [17])**.** *The universal enveloping algebra of the holonomy Lie algebra, $U(\mathfrak{h}_A)$, equals the linear strand, $R = \bigoplus_{p\geq 0} U^p_0$, of the Yoneda algebra $U = U(\mathfrak{g}_A)$.*

Particularly simple is the case when $A$ is a Koszul algebra. By definition, this means the homotopy Lie algebra $\mathfrak{g}_A$ coincides with the holonomy Lie algebra $\mathfrak{h}_A$, i.e., $U = R$. Alternatively, $A$ is quadratic (i.e., $A = \overline{A}$), and its quadratic dual, $A^! = T(V)/(I_2^{\perp})$, coincides with the Yoneda algebra: $A^! = U$. For an expository account of Koszul algebras, see [11].

As a simple (yet basic) example, take $E = \bigwedge V$, the exterior algebra on $V$. Then $E$ is Koszul, and its quadratic dual is $E^! = \mathrm{Sym}(V^*)$, the symmetric algebra on the dual vector space. Moreover, $\mathfrak{g}_A = \mathfrak{h}_A$ is the abelian Lie algebra on $V$.

1.2. **Main result.** The computation of the homotopy Lie algebra of a given algebra $A$ is, in general, a very hard problem. Our goal here is to determine $\mathfrak{g}_A$ under certain homological hypothesis. First, we need one more definition.

Let $B = \overline{A}$ be the quadratic closure of $A$. View $J = \ker(B \twoheadrightarrow A)$ as a graded left module over $B$.

**Definition 1.4.** The *homotopy module* of a graded algebra $A$ is

$$M_A = \mathrm{Ext}_B(J, \Bbbk), \tag{4}$$

viewed as a bigraded left module over the ring $R = U(\mathfrak{h}_A) = \mathrm{Ext}_B(\Bbbk, \Bbbk)$ via the Yoneda product.

**Theorem 1.5.** *Let $A$ be a graded algebra over a field $\Bbbk$, with quadratic closure $B = \overline{A}$, and homotopy module $M = M_A$. Assume $B$ is a Koszul algebra, and there exists an integer $\ell$ such that $M_q = 0$ unless $\ell \leq q \leq \ell + 1$. Then, as graded Hopf algebras,*

$$U(\mathfrak{g}_A) \cong T(M_A[-2]) \mathbin{\widehat{\otimes}}_{\Bbbk} U(\mathfrak{h}_A), \tag{5}$$

*where $M[q]^r = M^{q+r}$, and the action of $U(\mathfrak{h})$ on the tensor algebra of $M[-2]$ is induced from the $U(\mathfrak{h})$-module structure of $M[-2]$.*

Taking the Lie algebras of primitive elements in the respective Hopf algebras, we obtain the following.

**Corollary 1.6.** *Under the above hypothesis, the homotopy Lie algebra of $A$ splits as a semi-direct product of the holonomy Lie algebra with the free Lie algebra on the (shifted) homotopy module,*

$$\mathfrak{g}_A \cong \mathrm{Lie}(M_A[-2]) \rtimes \mathfrak{h}_A, \tag{6}$$

*where the action of $\mathfrak{h}$ on $\mathrm{Lie}(M)$ is given by $[m,h] = -hm$ for $h \in \mathfrak{h}$ and $m \in M$.*

1.3. **Hyperplane arrangements.** Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be an arrangement of hyperplanes in $\mathbb{C}^\ell$, with intersection lattice $L(\mathcal{A})$ and complement $X(\mathcal{A})$. The cohomology ring $A = H^\bullet(X(\mathcal{A}), \Bbbk)$ admits a combinatorial description (in terms of $L(\mathcal{A})$), due to Orlik and Solomon:

$$A = E/I, \tag{7}$$

where $E$ is the exterior algebra over $\Bbbk$, on generators $e_1, \dots, e_n$ in degree 1, and $I$ is the ideal generated by all elements of the form $\sum_{q=1}^{r}(-1)^{q-1} e_{i_1} \cdots \widehat{e_{i_q}} \cdots e_{i_r}$ for which $\mathrm{rk}(H_{i_1} \cap \cdots \cap H_{i_r}) < r$; see [20].

The holonomy Lie algebra of the Orlik-Solomon algebra also admits an explicit presentation, this time solely in terms of $L_{\le 2}(\mathcal{A})$. Identify $\mathrm{Lie}(A_1^*)$ with the free Lie algebra over $\Bbbk$, on generators $x_H = e_H^*$, $H \in \mathcal{A}$. Then:

$$\mathfrak{h}_A = \mathrm{Lie}(A_1^*) \Big/ \text{ideal}\Big\{ \Big[ x_H, \sum_{H' \in \mathcal{A}\colon H' \supset F} x_{H'} \Big] \mid F \in L_2(\mathcal{A}) \text{ and } F \subset H \Big\}. \tag{8}$$

As we shall see in Section 5, the homotopy Lie algebra $\mathfrak{g}_A$ also admits a finite presentation, for a certain class of hypersolvable arrangements, to be defined below.

**Question 1.7.** *Do there exist arrangements for which $\mathfrak{g}_A$ is not finitely presented? For which the (bigraded) Hilbert series of $U(\mathfrak{g}_A)$ is not a rational function?*

1.4. **Hypersolvable arrangements.** An arrangement $\mathcal{A}$ is called *supersolvable* if its intersection lattice admits a maximal modular chain. The OS algebra of a supersolvable arrangement has a quadratic Gröbner basis, and thus, it is a Koszul algebra (this result, implicit in Björner and Ziegler [2], was proven in Shelton and Yuzvinsky [28]).

An arrangement $\mathcal{A}$ is called *hypersolvable* if it has the same intersection lattice up to rank 2 as that of a supersolvable arrangement. This "supersolvable deformation," $\mathcal{B}$, is uniquely defined, and has the property that the two complements have isomorphic fundamental groups; see Jambu and Papadima [14, 15]. Let $A = H^\bullet(X(\mathcal{A}), \Bbbk)$ and $B = H^\bullet(X(\mathcal{B}), \Bbbk)$ be the respective OS algebras. It is readily seen that $B = \overline{A}$; thus, $A$ and $B$ share the same holonomy Lie algebra: $\mathfrak{h} = \mathfrak{h}_A = \mathfrak{h}_B$. Furthermore, since $B$ is Koszul, we have $\mathfrak{g}_B = \mathfrak{h}$.

The hypothesis of Theorem 1.5 holds in two nice situations, which can be checked combinatorially; for precise definitions, see §4.2 and §4.3, respectively.

**Theorem 1.8.** *Let $\mathcal{A}$ be an arrangement, and let $A$ be its Orlik-Solomon algebra. Suppose either*

(1) *$\mathcal{A}$ is hypersolvable, and its singular range has length* 0 *or* 1*; or*

(2) *$\mathcal{A}$ is obtained by fibred extensions of a generic slice of a supersolvable arrangement.*

*Then* $\mathfrak{g}_A \cong \mathrm{Lie}(M_A[-2]) \rtimes \mathfrak{h}_A$.

An explicit finite presentation for $\mathfrak{g}_A$ is given in Theorem 5.1, in the case when $\mathcal{A}$ is a generic slice of a supersolvable arrangement. The Eisenbud-Popescu-Yuzvinsky resolution [5] permits us to compute the Hilbert series of $M_A$ (and hence, that of $\mathfrak{g}_A$) in the case when $\mathcal{A}$ is a 2-generic slice of a Boolean arrangement.

Theorem 1.8 allows us to distinguish between hyperplane arrangements whose holonomy Lie algebras are isomorphic. In Example 6.2, we exhibit a pair of 2-generic, 4-dimensional sections of the Boolean arrangement in $\mathbb{C}^7$; the two arrangements have the same fundamental group, the same Poincaré polynomial, and the same holonomy Lie algebra, yet different homotopy Lie algebras.

In Section 7, we provide some topological interpretations. As noted in [3], [22], the holonomy Lie algebra of a supersolvable arrangement equals, up to a rescaling factor, the topological homotopy Lie algebra of the corresponding "redundant" subspace arrangement. We extend this result, and relate the homotopy Lie algebra of an arbitrary hyperplane arrangement to the topological homotopy Lie algebras of the redundant subspace arrangements. As a consequence, we find a pair of codimension-2 subspace arrangements in $\mathbb{C}^8$, whose complements are simply-connected and have the same homology groups, yet distinct higher homotopy groups.

## 2. Some homological algebra

2.1. **The homotopy module.** Let $A$ be graded, graded-commutative, connected, locally finite algebra. Assume $A$ is generated in degree 1, and its quadratic closure, $B = \overline{A}$ is a Koszul algebra. Let $E$ be the exterior algebra on $A_1 = B_1$. Let $I$ and $J$ be, respectively, the kernels of the natural surjections $E \twoheadrightarrow B$ and $B \twoheadrightarrow A$, giving the exact sequences

$$0 \longrightarrow I \longrightarrow E \longrightarrow B \longrightarrow 0\ , \tag{9}$$

$$0 \longrightarrow J \longrightarrow B \longrightarrow A \longrightarrow 0\ . \tag{10}$$

In what follows, we will record some homological properties of the ring $A$, viewed as a $B$-module. Recall if $N$ is a $B$-module, the Yoneda product gives $\mathrm{Ext}_B(N, \Bbbk)$ the structure of a left module over the ring $R = U(\mathfrak{h}_A) = \mathrm{Ext}_B(\Bbbk, \Bbbk)$. An object of primary interest for us will be the *homotopy module* of $A$,

$$M = M_A = \mathrm{Ext}_B(J, \Bbbk). \tag{11}$$

This bigraded $R$-module will play a crucial role in the determination of the homotopy Lie algebra $\mathfrak{g}_A$.

Our grading conventions shall be as follows. Suppose $V$ and $W$ are $\mathbb{Z}$-graded $\Bbbk$-vector spaces. Then $f \in \mathrm{Hom}_\Bbbk(V, W)$ has degree $r$ if $f\colon V^q \to W^{q+r}$ for all $q$. For any $\mathbb{Z}$-graded $\Bbbk$-vector space $V$, we shall let $V^*$ denote the graded $\Bbbk$-dual of $V$. In particular, then, $(V^*)^q = \mathrm{Hom}_\Bbbk(V^{-q}, \Bbbk)$. If $V$ has finite $\Bbbk$-dimension in each graded piece, then $(V^*)^* \cong V$.

We shall treat all boundary maps in chain complexes as having polynomial degree 0 and homological degree +1. Then, in particular, chain complexes will be regarded as cochain complexes in negative degree. We shall indicate shifts of polynomial grading by defining $V(q)^r = V^{q+r}$, and shifts of homological grading by writing $V[q]$ analogously.

Following these conventions, $M^{pq} = \mathrm{Ext}^p_B(J, \Bbbk)^q$ is nonzero only for $q \leq -p$. Then, taking $M^p_q = M^{p,-p-q}$ (the internal grading), we have $M^p_q \neq 0$ only for $q \geq 0$. The grading is such that, for each fixed $q$, the action of $R$ on $M$ satisfies $R^r \otimes M^p_q \to M^{r+p}_q$.

**Lemma 2.1.** $\mathrm{Ext}_B(A, \Bbbk) \cong \Bbbk \oplus M[-1]$ *as graded $R$-modules.*

*Proof.* Consider the long exact sequence for $\mathrm{Ext}_B(-, \Bbbk)$ applied to (10):

$$\cdots \longrightarrow \mathrm{Ext}^{q-1}_B(J, k) \longrightarrow \mathrm{Ext}^q_B(A, k) \longrightarrow \mathrm{Ext}^q_B(B, k) \longrightarrow \cdots \tag{12}$$

Since $\mathrm{Ext}^0_B(A, \Bbbk) \cong \mathrm{Ext}^0_B(B, \Bbbk) \cong \Bbbk$ and $\mathrm{Ext}^q_B(B, \Bbbk) = 0$ for all $q > 0$, the map $\mathrm{Ext}_B(B, \Bbbk) \to \mathrm{Ext}_B(J, \Bbbk)$ is zero. So the long exact sequence breaks into short exact sequences which, using (11), we will write as a single short exact sequence of graded $R$-modules,

$$0 \longrightarrow M[-1] \longrightarrow \mathrm{Ext}_B(A, \Bbbk) \longrightarrow \Bbbk \longrightarrow 0\ . \tag{13}$$

For each $q$, one of the two maps is zero and the other is an isomorphism, so the short exact sequence splits. □

2.2. **Injective resolutions.** For any $E$-module $N$, let

$$N^\circ = \{a \in E : ax = 0 \text{ for all } x \in N\}, \tag{14}$$

the annihilator of $N$ in $E$. Later on, we require explicit, injective resolutions.

**Lemma 2.2.** *Suppose the ring $B = E/I$ is an arbitrary quotient of a finitely-generated exterior algebra $E$. If*

$$0 \longleftarrow \Bbbk \longleftarrow B \otimes_\Bbbk F^0 \longleftarrow B \otimes_k F^1 \longleftarrow \cdots \tag{15}$$

*is a minimal, free resolution of $\Bbbk$ over $B$, then*

$$0 \longrightarrow \Bbbk \longrightarrow B^* \otimes_\Bbbk (F^0)^* \longrightarrow B^* \otimes_k (F^1)^* \longrightarrow \cdots \tag{16}$$

*is an injective resolution of $\Bbbk$ over $B$.*

*Proof.* The resolution (15) is an acyclic complex of $E$-modules, so its vector space dual (16) is an acyclic complex as well, since each $F^i$ has finite $\Bbbk$-dimension.

Now $B^* \cong I^\circ(n)$ as $E$-modules, via the determinantal pairing in $E$. On the other hand, $E$ is injective as a module over itself, so $I^\circ$ is injective as an $E$-module; see [27, Prop. 2.27]. Since each $F^i$ has finite $\Bbbk$-dimension, each $B^* \otimes_\Bbbk (F^i)^*$ is injective. □

**Lemma 2.3.** *Let $A$ and $B$ two algebras, with $\overline{A} = B$ Koszul. Write $B = E/I$, $A = B/J$, $\mathfrak{h} = \mathfrak{h}_A = \mathfrak{h}_B$, and $R = U(\mathfrak{h})$. Then:*

(1) *The complex*

$$0 \longrightarrow \Bbbk \longrightarrow I^\circ(n) \otimes_\Bbbk R^0 \longrightarrow I^\circ(n) \otimes_\Bbbk R^1 \longrightarrow \cdots \tag{17}$$

*is an injective resolution of* $\Bbbk$ *over* $B$*, with boundary map described below.*

(2) $\operatorname{Ext}^q_B(A, \Bbbk) \cong H^q(J^\circ(n) \otimes_\Bbbk R^\bullet)$*, for all* $q \geq 0$.

*Proof.* The Koszul complex $\mathcal{K}^* = B \otimes_\Bbbk R^*$ is a free $B$-module resolution of $\Bbbk$, so it is also an acyclic complex of $E$-modules, with boundary map induced from

$$\partial^* \colon 1 \otimes x_i^* \mapsto e_i \otimes 1. \tag{18}$$

Then $\operatorname{Hom}_\Bbbk(B \otimes_\Bbbk R^*, \Bbbk) = B^* \otimes_\Bbbk R$ is an injective resolution, by the previous Lemma.

To establish (2), it suffices to note that $\operatorname{Hom}_B(A, I^\circ) \cong J^\circ$. ☐

## 3. Proof of the main result

Our approach to the proof of Theorem 1.5 is to construct a spectral sequence comparing the minimal resolution and the Koszul complex of $A$. We show the spectral sequence collapses at $E_2$ under suitable hypotheses in Proposition 3.2, though not in general (Example 3.3). This collapsing is enough to prove the theorem, via Proposition 3.1.

3.1. **A spectral sequence.** Using the previous notation, $A \otimes_\Bbbk U^* \to \Bbbk \to 0$ is a minimal free resolution of $\Bbbk$ over $A$. It is filtered by degree, and the linear strand is $A \otimes_\Bbbk R^*$. That is, there is a short exact sequence of chain complexes

$$0 \longrightarrow A \otimes_\Bbbk R^* \xrightarrow{1\otimes\varepsilon^*} A \otimes_\Bbbk U^* \longrightarrow A \otimes_\Bbbk U^*_+ \longrightarrow 0\,. \tag{19}$$

Now $B \otimes_\Bbbk R^*$ is a free resolution of $\Bbbk$ over $B$, since $B$ is Koszul. Using Lemma 2.1, we find that the homology of the linear strand (Koszul complex) is

$$\begin{aligned} H_\bullet(A \otimes R^*) &\cong \operatorname{Tor}^B(A, \Bbbk) \\ &\cong \operatorname{Ext}_B(A, \Bbbk)^* \\ &\cong \Bbbk \oplus M[-1]^*. \end{aligned} \tag{20}$$

The long exact sequence in homology then reveals that

$$H_\bullet(A \otimes_\Bbbk U^*_+) \cong M[-2]^* \tag{21}$$

as $A$-modules. Recall that $A$ acts trivially on $M$ (and hence on $M[-2]^*$), so

$$\operatorname{Hom}_A(H_\bullet(A \otimes_\Bbbk U^*_+), \Bbbk) \cong M[-2]. \tag{22}$$

On the other hand, since our complex is a quotient of a minimal resolution,

$$H_\bullet(\operatorname{Hom}_A(A \otimes_\Bbbk U^*_+, \Bbbk)) \cong U_+. \tag{23}$$

Comparing the two gives a universal coefficients spectral sequence of the form

$$E_2^{pq} = \operatorname{Ext}^p_A((M[-2]^*)^q, \Bbbk) \cong M[-2]^q \otimes_\Bbbk U^p \implies U_+^{p+q}. \tag{24}$$

The spectral sequence is used as follows.

**Proposition 3.1.** *If $E_\infty = E_2$ in the spectral sequence (24), then*

$$0 \longrightarrow U \otimes_{\Bbbk} M[-2] \xrightarrow{\phi} U \xrightarrow{\varepsilon} R \longrightarrow 0$$

*is exact, and the conclusion of Theorem 1.5 holds.*

*Proof.* If $E_\infty = E_2$, then $U \otimes M[-2] \cong U_+$ as a (left) $U$-module. Now $U_+ = \ker \varepsilon$, giving the short exact sequence. Since $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$, $R = U(\mathfrak{h})$ is a Hopf subalgebra of $U = U(\mathfrak{g})$, so the sequence splits. The isomorphism of Theorem 1.5 can then be obtained by induction. $\square$

3.2. **Collapsing conditions.** In order to show that the higher differentials in the spectral sequence (24) vanish, we use a degree argument that begins by considering the $E_0$ term. Since

$$0 \longrightarrow \Bbbk \longrightarrow A^* \otimes_{\Bbbk} U^0 \longrightarrow A^* \otimes_{\Bbbk} U^1 \longrightarrow \cdots \tag{25}$$

is an injective resolution of $\Bbbk$ over $A$, (Lemma 2.2) we consider the double complex

$$\begin{aligned} C^{pq} &= \operatorname{Hom}_A(A \otimes_{\Bbbk} (U^q)^*_+, A^* \otimes_{\Bbbk} U^p) \\ &\cong U^q_+ \otimes_{\Bbbk} A^* \otimes_{\Bbbk} U^p, \end{aligned} \tag{26}$$

with induced boundary maps $\partial_h$ and $\partial_v$. Then our spectral sequence (24) is obtained by filtering $C^{\bullet\bullet}$ by columns. Checking the grading, we see

$$\partial_v \colon U^q_+ \otimes_{\Bbbk} (A^*)^s \otimes_{\Bbbk} U^p \to U^{q+1}_+ \otimes_{\Bbbk} (A^*)^{s+1} \otimes_{\Bbbk} U^p \tag{27}$$

and

$$\partial_h \colon U^q_+ \otimes_{\Bbbk} (A^*)^s \otimes_{\Bbbk} U^p \to U^q_+ \otimes_{\Bbbk} (A^*)^{s+1} \otimes_{\Bbbk} U^{p+1}. \tag{28}$$

By looking at $E_2$ and $\partial_v$, we see that we must have $E_1 = E_2$.

We first consider the case where the ideal $J$ has a (shifted) linear resolution.

**Proposition 3.2.** *Suppose $\mathcal{A}$ is a hypersolvable arrangement for which $M^p_q = 0$ unless $q = \ell$, for some fixed $\ell$. Then $E_2 = E_\infty$.*

*Proof.* In this case, $M[-2]^q_r = 0$ unless $r = \ell - 2$. Then $H^q(C^{p\bullet}, \partial_v)_r = 0$ unless $r = \ell - 2$.

First we note that $(U_+)^q_t = 0$ unless $t \geq \ell - 2$. This can be seen from the fact that $U_+$ is a graded subquotient of $M[-2] \otimes_{\Bbbk} U$, from (24): the support of $M[-2]$ is described above, and $U^p_q = 0$ unless $q \geq 0$.

Regard $A^*$ as a chain complex concentrated in homological degree 0. Then observe that the internal degree of a nontrivial cocycle representative in $(U_+)^q_t \otimes_{\Bbbk} (A^*)_s$ is $s + t = \ell - 2$, by the first observation above. It follows $s \leq 0$ from the inequality above. However, $(A^*)_s = 0$ unless $0 \leq s \leq \ell$, so the representative of a nonzero, homogeneous $E_2$-cocycle in $E_0$ must have $s = 0$.

Now suppose $x \in E^{pq}_2$ is such a cocycle, with representative $\widetilde{x}$ in $C^{pq}$. By the above, $\widetilde{x} \in U^q_+ \otimes_{\Bbbk} (A^*)_0$. Then $\partial_h(\widetilde{x}) = 0$ in $C^{p+1,q}$ by (28). This means $d_2(x) = 0$, and similarly for higher differentials. $\square$

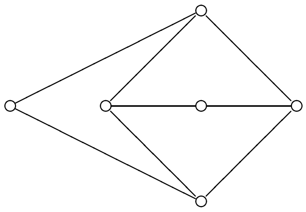
*Proof of Theorem 1.5.* In view of Proposition 3.1, it remains only to show the spectral sequence collapses when $M^p_q = 0$ unless $0 \leq \ell - q \leq 1$ for some $\ell$. In this case, let $N = M_\ell$ denote the $R$-submodule of $M$ of internal degree $\ell$.

By the same reasoning as in the proof of Proposition 3.2, $N[-2] \otimes U \subseteq \ker d_k$ for $k \geq 2$. Now $N[-2] \cong N[-2] \otimes U^0$ is a submodule of the $p = 0$ column of $E_2$. Since it is (trivially) not in the image of any nonzero differentials, $N[-2]$ is an $R$-submodule of $U$.

Let $K$ denote the Hopf subalgebra of $U$ generated by $R$ and $N[-2]$. By [18, Theorem 4.4], $U$ is a free $K$-algebra. It follows that $K \cong T(N[-2]) \otimes_{\Bbbk} R$. In the notation of the previous proposition, any nontrivial differential $d_k$ with $k \geq 2$ would lift in $E_0$ to a map $U_+ \otimes (A^*)_1 \otimes U \to U_+ \otimes (A^*)_0 \otimes U$. We have shown that the targets of these maps are unchanged between $E_2$ and $E_\infty$, so it follows that the maps themselves must also all be zero. □

3.3. **A non-collapsing spectral sequence.** Calculations with the Macaulay 2 package [13] show that the hypotheses of Theorem 1.5 cannot in general be relaxed: differentials in the spectral sequence (24) may not be zero.

**Example 3.3.** Consider the graphic arrangement associated with the following graph,



Let $A$ be the Orlik-Solomon algebra, and $M = M_A$ its homotopy module. It is readily seen that $M_q \neq 0$ for $q = 3, 4, 5$. An Euler characteristic calculation shows that the spectral sequence (24) must have a nonzero differential

$$d_2^{04}\colon M[-2]^4_6 \otimes_{\Bbbk} U^0 \to M[-2]^3_5 \otimes_{\Bbbk} U^2.$$

It follows that the Hopf algebra $U(\mathfrak{g}_A)$ will not have the structure we find in Theorem 1.5.

## 4. Hypersolvable arrangements

In this section, we apply our main result to certain classes of hypersolvable arrangements.

4.1. **Solvable extensions.** We start by reviewing in more detail the notion of a hypersolvable arrangement, introduced by Jambu and Papadima in [14]. Roughly, a hypersolvable arrangement is a linear projection of a supersolvable arrangement that preserves intersections through codimension two.

**Definition 4.1** ([14])**.** An arrangement $\mathcal{A}$ is hypersolvable if there exist subarrangements $\{0\} = \mathcal{A}_1 \subset \mathcal{A}_2 \subset \cdots \subset \mathcal{A}_m = \mathcal{A}$, so that each inclusion $\mathcal{A}_i \subset \mathcal{A}_{i+1}$ is *solvable.* In turn, an inclusion of hyperplane arrangements $\mathcal{A} \subset \mathcal{B}$ is called a solvable extension if:

(1) There are no hyperplanes $H \in \mathcal{B} \setminus \mathcal{A}$ and $H', H'' \in \mathcal{A}$ with $H' \neq H''$ and $\operatorname{rk}(H \cap H' \cap H'') = 2$;
(2) For any $H, H' \in \mathcal{B} \setminus \mathcal{A}$, there is exactly one $H'' \in \mathcal{A}$ with $\operatorname{rk}(H \cap H' \cap H'') = 2$, denoted by $f(H, H')$;
(3) For any $H, H', H'' \in \mathcal{B} \setminus \mathcal{A}$, one has $\operatorname{rk}(f(H, H') \cap f(H, H'') \cap f(H', H'')) \leq 2$.

It turns out that if $\mathcal{A}$ is hypersolvable with a sequence of solvable extensions as above, then for all $i$, the rank of $\mathcal{A}_i$ and $\mathcal{A}_{i+1}$ differ by at most one. If the ranks are equal, the extension is said to be singular; otherwise, the extension is nonsingular (or fibred, in the sense of Falk and Randell, [9]).

If $s$ denotes the number of singular extensions, then, $\operatorname{rk} \mathcal{A} = m - s$. Jambu and Papadima show in [15] that one can replace the singular extensions by nonsingular ones in order to construct a supersolvable arrangement $\mathcal{B}$ of rank $m$ that projects onto $\mathcal{A}$, preserving the intersection lattice through rank 2. That is,

**Theorem 4.2.** *An arrangement $\mathcal{A}$ is hypersolvable iff there exists a supersolvable arrangement $\mathcal{B}$ and a linear subspace $W$ for which $\mathcal{A} = \mathcal{B} \cap W$ and $L(\mathcal{A})_{\leq 2} \cong L(\mathcal{B})_{\leq 2}$.*

*Proof.* The implication "⇒" is Theorem 2.4 of [15]. The converse, due to Jambu (private communication), runs as follows. Suppose $\mathcal{B}$ is supersolvable and there exists a subspace $W$ as above. By definition, $\mathcal{B}$ has a maximal modular chain $F_1 < F_2 < \cdots < F_m$. Putting $\mathcal{B}_i = \mathcal{B}_{X_i}$ gives a sequence of solvable extensions for $\mathcal{B}$, all fibred. For $1 \leq i \leq m$, let $\mathcal{A}_i = \mathcal{B}_i \cap W$. Since collinearity relations are preserved, each $\mathcal{A}_i \subset \mathcal{A}_{i+1}$ is also a solvable extension, so $\mathcal{A}$ is hypersolvable. □

We remark that, in the above proof, $\mathcal{A}_i \subset \mathcal{A}_{i+1}$ is a singular extension if and only if $F_i \cap W = F_{i+1} \cap W$. The arrangement $\mathcal{B}$ in called the *supersolvable deformation* of $\mathcal{A}$. For example, any arrangement $\mathcal{A}$ for which no three hyperplanes intersect in codimension three is hypersolvable, and its supersolvable deformation is the Boolean arrangement in $\mathbb{C}^n$, where $n = |\mathcal{A}|$.

**Lemma 4.3.** *Suppose $\mathcal{A}' \subset \mathcal{A}$ is a fibred extension. The projection $p\colon X(\mathcal{A}) \to X(\mathcal{A}')$ induces an inclusion $A' \hookrightarrow A$ of the respective Orlik-Solomon algebras which makes $A$ into a free $A'$-module of rank $k = |\mathcal{A} \setminus \mathcal{A}'|$.*

*Proof.* The projection $p\colon X \to X'$ is a bundle map, with fiber $\mathbb{C} \setminus \{k \text{ points}\}$. As noted by Falk and Randell [9], this bundle admits a section, and thus the Serre spectral sequence collapses at the $E_2$ term. Hence, $H^\bullet(X) \cong H^\bullet(X') \otimes H^\bullet(\vee^k S^1)$. The result follows. □

4.2. **Singular range.** We now give some easy to check combinatorial conditions insuring that a hypersolvable arrangement satisfies the hypothesis of Theorem 1.5. We start by attaching a pair of relevant integers to a hypersolvable arrangement.

**Definition 4.4.** Suppose $\mathcal{A}$ is hypersolvable with supersolvable deformation $\mathcal{B}$, and $\mathcal{A} \neq \mathcal{B}$. Let $c$ be the least integer for which $L(\mathcal{A})_{\leq c} \ncong L(\mathcal{B})_{\leq c}$. Since $\mathcal{A} \neq \mathcal{B}$, there is a largest integer $i$ for which the extension $\mathcal{A}_i \subset \mathcal{A}_{i+1}$ is singular. Let $d$ the rank of these two arrangements. We will call the pair $(c, d)$ the *singular range* of the arrangement $\mathcal{A}$, and $|d - c|$ the length of this range.

**Lemma 4.5.** *If $\mathcal{A}$ is hypersolvable with singular range $(c, d)$, then $3 \le c \le d$.*

*Proof.* The inequality $c \geq 3$ follows from Theorem 4.2. Suppose $d < c$; then $L(\mathcal{A})_{\le d} \cong L(\mathcal{B})_{\le d}$. It follows that $L(\mathcal{A}_{d+1})_{\le d} \cong L(\mathcal{B}_{d+1})_{\le d}$, whence $\mathcal{A}_{d+1} = \mathcal{B}_{d+1}$ since the arrangements are central. Since $d$ is greater than or equal to the index of the last singular extension, however, $\mathcal{A}_i = \mathcal{B}_i$ for $d+1 \le i \le m$, so $\mathcal{A} = \mathcal{B}$, a contradiction. □

Let $A = H^\bullet(X(\mathcal{A}), \Bbbk)$ and $B = H^\bullet(X(\mathcal{B}), \Bbbk)$ be the respective Orlik-Solomon algebras. Since $L(\mathcal{A})_{\le 2} \cong L(\mathcal{B})_{\le 2}$, and since the Orlik-Solomon algebra of a supersolvable arrangement is quadratic, the algebra $B = E/I$ is the quadratic closure of $A$. Let $J = \ker(B \twoheadrightarrow A)$, and let $M = \operatorname{Ext}_B(J, \Bbbk)$, viewed as a module over $R = \operatorname{Ext}_B(\Bbbk, \Bbbk)$. Since $\mathcal{B}$ is supersolvable, the algebra $B$ is Koszul (see [28]); thus, $R = B^!$.

**Lemma 4.6.** *If $\mathcal{A}$ is a hypersolvable arrangement with singular range $(c, d)$, then $M^p_q = 0$ unless $p \ge 0$ and $c \le q \le d$.*

*Proof.* The ideal $J$ has a minimal, (infinite) free resolution over $B$ of the form

$$0 \longleftarrow J \longleftarrow B \otimes_{\Bbbk} (M^{0,-})^* \longleftarrow B \otimes_{\Bbbk} (M^{1,-})^* \longleftarrow \cdots \tag{29}$$

Recall that $J$ is generated by Orlik-Solomon relations. By Definition 4.4, the least degree of a generator of $J$ is $c$, so $M^0_c \neq 0$ and $M^0_q = 0$ for $q < c$. Thus $M^p_q = 0$ for $q < c$, establishing the first inequality.

To show $M^p_q = 0$ for $q > d$, too, let $i$ be the largest index of a singular extension $\mathcal{A}_i \subset \mathcal{A}_{i+1}$. let $B_{i+1} = H^\bullet(X(\mathcal{B}_{i+1}), \Bbbk)$ and $A_{i+1} = H^\bullet(X(\mathcal{A}_{i+1}), \Bbbk)$, and let $B'_{i+1} = H^\bullet(X(\mathcal{B}'_{i+1}), \Bbbk)$ be the cohomology ring of the projectivization (decone) of $\mathcal{B}_{i+1}$. Recall from [20] that $X(\mathcal{B}_{i+1}) = X(\mathcal{B}'_{i+1}) \times \mathbb{C}^\times$. From the Künneth formula, we obtain the following exact sequence of $B'_{i+1}$-modules:

$$0 \longrightarrow B'_{i+1} \longrightarrow B_{i+1} \longrightarrow B'_{i+1}(-1) \longrightarrow 0. \tag{30}$$

Let $J_{i+1}$ denote the kernel of the canonical projection $B_{i+1} \twoheadrightarrow A_{i+1}$. If we let $J' = J_{i+1} \cap B'_{i+1}$, then $J_{i+1} = B_{i+1} \otimes_{B'_{i+1}} J'$, as a module over $B_{i+1}$. Since $\mathcal{A}$, $\mathcal{B}$ are obtained from $\mathcal{A}_{i+1}$, $\mathcal{B}_{i+1}$, respectively, by a sequence of fibred extensions, $J = B \otimes_{B_{i+1}} J_{i+1}$.

On the other hand, $B_{i+1}$ is a free module over $B'_{i+1}$, and by applying Lemma 4.3 inductively, $B$ is free over $B_{i+1}$. Therefore, $B'_{i+1} \to B$ is a flat change of rings, and it is enough to check that

$$\operatorname{Ext}^p_{B'_{i+1}}(J', \Bbbk)_q = 0 \tag{31}$$

if $q > d$. By Lemma 2.1, $\operatorname{Ext}^p_{B'_{i+1}}(J', \Bbbk)_q = \operatorname{Ext}^{p+1}_{B'_{i+1}}(A'_{i+1}, \Bbbk)_{q-1}$. Since $B'_{i+1}$ is Koszul and $(A'_{i+1})_q = 0$ for $q > d - 1$, the rank of the arrangement, the groups (31) are zero for $q > d$ by [16, Lemma 2.2]. □

The Lemma says, in particular, that the $B$-module $J(-c)$ has Castelnuovo-Mumford regularity no greater than the length of the singular range, $d - c$. Moreover, the Lemma gives a combinatorial condition for the hypotheses of Theorem 1.5 to be satisfied.

**Corollary 4.7.** *If $\mathcal{A}$ is hypersolvable and its singular range has length* 0 *or* 1*, then* $\mathfrak{g}_A \cong \mathrm{Lie}(M[-2]) \rtimes \mathfrak{h}_A$.

**Example 4.8** (2-generic arrangements of rank 4)**.** Suppose $\mathcal{A}$ is a central arrangement in $\mathbb{C}^4$, with the property that no three hyperplanes contain a common plane. Such an arrangement is hypersolvable, by Theorem 4.2, with supersolvable deformation $\mathcal{B}$ a Boolean arrangement. From Definition 4.4 and Lemma 4.5, $3 \leq c \leq d \leq 4$, so the singular range has length 0 or 1.

On the other hand, the graphic arrangement from Example 3.3 is hypersolvable, with singular range $(3, 5)$, and Corollary 4.7 does not apply (indeed, its conclusion fails).

4.3. **Generic slices of supersolvable arrangements.** Lemma 4.6 provides bounds on the polynomial degrees of the homotopy module $M$, which cannot be improved without imposing further restrictions on the arrangement. In general, it is not obvious how to characterize the support of $M$ combinatorially; the problem seems similar to that of characterizing which arrangements have quadratic defining ideals, investigated in particular in [8, 4]. To this end, we isolate a class of hypersolvable arrangements for which the situation is more manageable.

**Definition 4.9.** A codimension-$k$ linear space $W$ is said to be generic with respect to an arrangement $\mathcal{B}$ if $\mathrm{rk}(X \cap W) = \mathrm{rk}\, X + k$ for all $X \in L(\mathcal{B})$ with $\mathrm{rk}\, X \leq \mathrm{rk}\, \mathcal{B} - k$.

If $\mathcal{B}$ is an essential, supersolvable arrangement of rank $m$ and $W$ is a proper, linear space of dimension $\ell \geq 3$, then by Theorem 4.2, the arrangement $\mathcal{A} = \mathcal{B} \cap W$ is hypersolvable. We call such an arrangement a *generic (hypersolvable) slice* of rank $\ell$.

Not every hypersolvable arrangement is a generic slice, see Example 4.15 from [21].

**Lemma 4.10.** *Let $\mathcal{B}$ be a rank $m$ supersolvable arrangement, and let $\mathcal{A}$ be a rank $\ell$ generic slice. Then the singular range of $\mathcal{A}$ is $(\ell, \ell)$.*

*Proof.* The assumption of genericity means $L(\mathcal{A})_{\leq \ell-1} \cong L(\mathcal{B})_{\leq \ell-1}$. However, $X \cap W = 0$ for all $X \in L(\mathcal{B})_\ell$, so since $W$ is proper and $\mathcal{B}$ is essential, the singular range of $\mathcal{A}$ is $(\ell, d)$ for some $d$. On the other hand, $\mathrm{rk}\, \mathcal{A}_\ell = \mathrm{rk}\, \mathcal{A}_m = \ell$, so the last $m - \ell$ extensions are all singular, and $d = \ell$. □

This is to say that, for generic slice arrangements, the module $J(-\ell)$ has a linear resolution. Slightly more generally:

**Proposition 4.11.** *Let $\mathcal{A}$ be a rank $\ell$ hypersolvable arrangement. Suppose there exists a generic slice $\mathcal{C}$ and fibred extensions $\mathcal{C} = \mathcal{A}_{m-i} \subset \cdots \subset \mathcal{A}_{m-1} \subset \mathcal{A}_m = \mathcal{A}$, for some $i \geq 0$. Then the singular range of $\mathcal{A}$ is $(\ell, \ell)$.*

*Proof.* As in the proof of Lemma 4.6, we may reduce to the case where $\mathcal{A} = \mathcal{C}$, a generic slice of rank $\ell$. Let $\mathcal{B}$ be the supersolvable deformation of $\mathcal{A}$. Denote by $A'$ and $B'$ the Orlik-Solomon algebras of the respective decones, and let $J' = \ker(B' \twoheadrightarrow A')$.

Let $R' = (B')^!$, and let $\mathcal{K} = R' \otimes_{\Bbbk} (B')^*$ be the corresponding Koszul complex. That is, $\mathcal{K}^q = R'(-q) \otimes_{\Bbbk} (B'^q)^*$ for $q \geq 0$, with differential $\partial \colon e_i^* \otimes 1 \mapsto 1 \otimes x_i$. Since $B'$ is a Koszul algebra, $\mathcal{K}$ is a free resolution of $\Bbbk$ over $R'$.

Let $M'$ be the $\ell$-th syzygy module in the resolution $\mathcal{K} \to \Bbbk \to 0$. That is, $M'$ is the cokernel of $\partial_{\ell+1}$, a left $R'$-module, which means $M'$ has minimal free resolution

$$0 \longrightarrow \mathcal{K}^m \xrightarrow{\partial_m} \cdots \longrightarrow \mathcal{K}^{\ell+1} \xrightarrow{\partial_{\ell+1}} \mathcal{K}^{\ell} \xrightarrow{\eta} M' \longrightarrow 0. \tag{32}$$

From this we see that $M'$ is concentrated in internal degree $\ell$, and $\mathrm{Ext}_{R'}(M',\Bbbk) \cong J'$, as a $B'$-module. Since Koszul duality is an involution, $\mathrm{Ext}_{B'}(J',\Bbbk) \cong M'$ as a left $R'$-module, and $M'$ is bigraded as claimed. □

The Proposition gives another criterion for the hypotheses of Theorem 1.5 to be satisfied. We obtain:

**Corollary 4.12.** *If $\mathcal{A}$ is obtained by fibred extensions of a generic slice of a supersolvable arrangement, then $\mathfrak{g}_A \cong \mathrm{Lie}(M[-2]) \rtimes \mathfrak{h}_A$.*

4.4. **Hilbert series.** Expressions for the Hilbert series of the graded module $M = \mathrm{Ext}_B(J,\Bbbk)$ are not known in general: compare with [26]. However, a simple formula exists for generic slices, which can be extended to fibred extensions of generic slices.

Let $\beta_i$ denote the $i$th Betti number of $B'$, so that $h(B',t) = \sum_{i=0}^{m} \beta_i t^i$ is its Hilbert series. The following fact is well-known; see [20].

**Lemma 4.13.** *There exist positive integers $1 = d_1 \le d_2 \le \cdots \le d_m$ for which*

$$h(B',t) = \prod_{j=2}^{m} (1 + d_j t).$$

By taking the Euler characteristic of (32), we note that for a generic slice of dimension $\ell$,

$$h_R(M,t) = h_{R'}(M',t) = h(R',t) \sum_{i=0}^{m-\ell} (-1)^i \beta_{i+\ell} t^i. \tag{33}$$

More generally, a fibred extension results in the same formula. Under the hypotheses of Theorem 1.5, together with formula (33), we have:

**Corollary 4.14.** *If $h(U,t,u) = \sum_{p,q} \dim_{\Bbbk} U_q^p t^p u^q$ is the bigraded Hilbert series of $U = U(\mathfrak{g}_A)$, then*

$$h(U,t,u) = h(R,t) \left(1 - u^{-2} h_R(M,t,u)\right)^{-1}. \tag{34}$$

*In the case of a generic slice of dimension $\ell$,*

$$h(U,t,u) = h(R,t) \left(1 - t^2 u^{-2} h(R,t) \sum_{i=0}^{m-\ell} (-1)^i \beta_{i+\ell} t^i \right)^{-1}. \tag{35}$$

## 5. A presentation for the homotopy Lie algebra

For the hypersolvable arrangements satisfying the hypotheses of Theorem 1.8, the problem of writing an explicit presentation for the homotopy Lie algebra $\mathfrak{g}_A$ is equivalent to that of presenting the homotopy module $M_A = \mathrm{Ext}_B(J,\Bbbk)$. We carry out this computation for generic slices of supersolvable arrangements.

Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be a hypersolvable arrangement, with supersolvable deformation $\mathcal{B}$. As usual, let $\mathfrak{h}$ denote the holonomy Lie algebra, and $R = U(\mathfrak{h})$ its enveloping algebra. Recall $\mathfrak{h}$ has a presentation with $n$ generators $x_1, \dots, x_n$ in degree $(1, 0)$, one for each hyperplane $H_i \in \mathcal{A}$, and for each flat $F \in L_2(\mathcal{A}) = L_2(\mathcal{B})$, relations

$$[x_i, \sum_{j \in F} x_j] = 0, \tag{36}$$

for all $i$ for which $i \in F$ (i.e., $F \subset H_i$).

Now assume $\mathcal{A}$ is a generic slice of a supersolvable arrangement. Then the resolution (32) gives a presentation of the (deconed) homotopy module $M'$ as an $R'$-module. In order to use this presentation explicitly, we will choose the basis for $B'^*$ given by identifying it with the flag complex of $\mathcal{B}'$, for which we refer to [4].

Recall $\mathrm{Fl}_p$ is a free $\Bbbk$-module on "flags" $(F_1, \dots, F_p)$, where $F_i \in L_i(\mathcal{B}')$ for $1 \leq i \leq p$, and $F_i < F_{i+1}$, modulo the following relations:

$$\sum_{G\colon F_{i-1} < G < F_{i+1}} (F_1, \dots, F_{i-1}, G, F_{i+1}, \dots, F_p), \tag{37}$$

for each $i$, $1 < i < p$. Moreover, the map $f\colon \mathrm{Fl}_p \to (B'^p)^*$ given by

$$f\colon (F_1, \dots, F_p) \mapsto \Big(\sum_{i \in F_1} e_i^*\Big)\Big(\sum_{i \in F_2 - F_1} e_i^*\Big) \cdots \Big(\sum_{i \in F_p - F_{p-1}} e_i^*\Big), \tag{38}$$

is an isomorphism, cf. [25, dual of (2.3.2)].

Under the identification $\mathrm{Fl} \cong B'^*$, the boundary map in the Koszul complex becomes the following. Given a flag $\mathbf{F} = (F_1, \dots, F_p)$ and $i \in F_p$, define an element $\mathbf{F} - i \in \mathrm{Fl}^{p-1}$ by finding the integer $j$ for which $i \in F_j - F_{j-1}$, and letting

$$\mathbf{F} - i = (-1)^{j-1} \sum (F_1, \dots, F_{j-1}, G_j, G_{j+1}, \dots, G_{p-1}), \tag{39}$$

where the sum is taken over all flags with the property that $i \notin G_{p-1}$ and $G_k < F_{k+1}$ for all $k$, $j \leq k < p$. Then the boundary map is given by extending

$$\partial\colon (F_1, \dots, F_p) \mapsto \sum_{i \in F_p} (\mathbf{F} - i) \otimes x_i \tag{40}$$

$R$-linearly.

For each element $\mathbf{F} \in \mathrm{Fl}^\ell$, let $y_{\mathbf{F}}$ denote the corresponding element of $M'$; that is, $y_{\mathbf{F}} = \eta \circ (f \otimes 1)(\mathbf{F} \otimes 1)$. In particular, we find a minimal generating set for $M'$ by choosing a set of $\beta_\ell$ flags of length $\ell$ in $L(\mathcal{B})$ appropriately. In particular, one may construct a basis for $\mathrm{Fl}^\ell$ using **nbc**-sets: see, for example, [4, Lemma 3.2].

Then the relations in $M'$ are given by the image of $\partial_{\ell+1}$ in (32). We have, for each flag $\mathbf{F} = (F_1, \dots, F_{\ell+1})$, a relation in $M'$ of the form

$$\sum_{i \in F_{\ell+1}} y_{\mathbf{F} - i} x_i. \tag{41}$$

It follows that in $\mathfrak{g}_A$, for each flag $\mathbf{F} = (F_1, \dots, F_{\ell+1})$, we have a relation

$$\sum_{i\in F_{\ell+1}} [x_i, y_{\mathbf{F}-i}]. \tag{42}$$

Now $M'$ is the restriction of the module $M$ from $R$ to $R'$, so the above gives a presentation for $M$ as well, noting that the central element $\sum_{i=1}^n x_i$ in $R$ acts trivially. One can find a minimal set of relations just by taking the flags $\mathbf{F}$ above to come from a basis of $\mathrm{Fl}^{\ell+1}$. We summarize this discussion, as follows.

**Theorem 5.1.** *Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be a generic slice of a supersolvable arrangement, and let $A$ be the Orlik-Solomon algebra of $\mathcal{A}$. Then, the homotopy Lie algebra $\mathfrak{g}_A$ has presentation with generators*

- *$x_i$ in degree $(1, 0)$, for each $i \in [n]$,*
- *$y_{\mathbf{F}}$ in degree $(2, \ell-2)$, for each $\mathbf{F} \in \mathrm{Fl}^\ell$,*

*and relations*

- *$\big[x_i, \sum_{j\in F} x_j\big] = 0$, for each flat $F \in L_2(\mathcal{A})$ and each $i \in F$,*
- *$\sum_{i\in F_{\ell+1}} \big[x_i, y_{\mathbf{F}-i}\big] = 0$, for each flag $\mathbf{F} = (F_1, \dots, F_{\ell+2}) \in \mathrm{Fl}^{\ell+1}$,*
- *$\big[\sum_{i=1}^n x_i, y_{\mathbf{F}}\big] = 0$, for each $\mathbf{F} \in \mathrm{Fl}^\ell$.*

We illustrate the above with an example.

**Example 5.2.** Consider the arrangement $\mathcal{A}$ defined by the polynomial

$$Q_{\mathcal{A}} = xyz(x-z)(y-z)(2x-y-4z)(2x-y-5z)(x+5y+2z)(x+5y+z).$$

This is a generic slice of the supersolvable arrangement $\mathcal{B}$, the cone over the arrangement defined by the polynomial $Q_{\mathcal{B}'} = vwxy(x-1)(y-1)(v-1)(w-1)$. The Poincaré polynomials of the deconed arrangements are given by

$$\begin{aligned}\pi(\mathcal{A}', t) &= 1 + 8t + 24t^2,\\ \pi(\mathcal{B}', t) &= (1+2t)^4 = 1 + 8t + 24t^2 + 32t^3 + 16t^4.\end{aligned}$$

Thus the homotopy module $M'$ has 32 generators and 16 relations, which can be described as follows.

Label the hyperplanes of $\mathcal{B}'$ as $0_0, 1_0, 2_0, 3_0, 0_1, 1_1, 2_1, 3_1$, in the order above. A basis of 32 flags of length 3 can be constructed by choosing three intersecting hyperplanes $i_a, j_b, k_c$, with $0 \le i < j < k \le 3$ and $a, b, c \in \{0, 1\}$, and forming a flag by successively intersecting the hyperplanes, from right to left. We will call this flag $\mathbf{F}_{i_a j_b k_c}$. A basis of 16 flags of length 4 in $\mathcal{B}'$ is constructed by choosing four intersecting hyperplanes, $0_a, 1_b, 2_c, 3_d$, for all choices of $a, b, c, d \in \{0, 1\}$, and forming a flag again by successive intersection.

Let $\mathfrak{g}_A$ be the holonomy Lie algebra of $\mathcal{A}$. Then $\mathfrak{g}_A$ has one generator $x_H$ for each hyperplane $H$, together with 32 additional generators $y_{i_a j_b k_c}$ in degree $(2, 1)$, and relations

$$[x_{0_a}, y_{1_b 2_c 3_d}] - [x_{1_b}, y_{0_a 2_c 3_d}] + [x_{2_c}, y_{0_a 1_b 3_d}] - [x_{3_d}, y_{0_a 1_b 2_c}],$$

for each $a, b, c, d \in \{0, 1\}$, in addition to the holonomy relations (8), and relations

$$\Big[\sum_{H\in\mathcal{A}} x_H, y_{i_a j_b k_c}\Big]$$

for each choice of $i, j, k,\ a, b, c$.

## 6. Two-generic arrangements of rank four

We now present a method for computing the Hilbert series of the homotopy Lie algebra of a particularly nice class of arrangements: rank-4 arrangements for which no three hyperplanes contain a common plane.

For any rank $\ell$ arrangement $\mathcal{A}$ with $n$ hyperplanes, let $E = \bigwedge_{\Bbbk}(e_1, \dots, e_n)$ be the exterior algebra, $A = E/I$ the Orlik-Solomon algebra, and $S = \Bbbk[x_1, \dots, x_n]$ the polynomial algebra. We recall the following.

**Theorem 6.1** (Eisenbud-Popescu-Yuzvinsky [5]). *The complex of $S$-modules*

$$0 \longleftarrow F(\mathcal{A}) \longleftarrow A^\ell \otimes S \longleftarrow \cdots \longleftarrow A^1 \otimes S \longleftarrow A^0 \otimes S \longleftarrow 0$$

*is exact, where the boundary maps are induced by multiplication by $\sum_{i=1}^n e_i \otimes x_i$, and the $S$-module $F(\mathcal{A})$ is taken as the cokernel of the map $A^{\ell-1} \otimes S \to A^\ell \otimes S$.*

It follows from Bernstein-Gelfand-Gelfand duality that, for each $p \geq 0$, there is a graded isomorphism of $S$-modules,

$$\operatorname{Ext}^p_E(A, \Bbbk)_q = \operatorname{Ext}^{\ell-q}_S(F(\mathcal{A}), S)_{p+q}. \tag{43}$$

We refer to [26] for the case of the smallest $q > 0$ for which this is nonzero. Details will appear in further work.

Now let $\mathcal{A}$ be a 2-generic arrangement. Notice that $B = E$ and $U(\mathfrak{h}) = B^! = S$. Then, applying Lemma 2.1 to (43), we obtain

$$M^p_q = \operatorname{Ext}^{\ell-q+1}_S(F(\mathcal{A}), S)_{p+q}, \tag{44}$$

for $p \geq 0$ and $0 \leq q \leq \ell$. As a result, presentations for the $S$-modules $M_q$ can be obtained computationally for specific examples, using formula (44).

We recall from Example 4.8 that, if the rank of the arrangement $\ell = 4$, then $\mathcal{A}$ satisfies hypotheses (1) of Theorem 1.8: $M_q = 0$ unless $q = 3$ or $q = 4$, i.e., the singular range of $\mathcal{A}$ is $(3, 4)$.

**Example 6.2.** Consider arrangements $\mathcal{A}_1$ and $\mathcal{A}_2$ defined by the polynomials

$$Q_1 = xyzw(x + y + z)(y + z + w)(x - y + z + w),$$
$$Q_2 = xyzw(x + y + z)(y + z + w)(x - y + z - w).$$

Both arrangements have 7 hyperplanes and 5 lines that each contain 4 hyperplanes, so the characteristic polynomials are $\pi(\mathcal{A}_1, t) = \pi(\mathcal{A}_2, t) = 1 + 7t + 21t^2 + 30t^3 + 15t^4$. Since there are no nontrivial intersections in codimension 2, the fundamental group of both complements is $\mathbb{Z}^7$, and $R = U(\mathfrak{h})$ is a polynomial ring.

We now use (44) to compute the Hilbert series of the graded modules $M_3$ and $M_4$ (recalling $M_q = 0$ for $q \neq 3, 4$). With the help of Macaulay 2, we find for $\mathcal{A}_1$

$$h(M_3, t) = (5 + 2t)/(1 - t)^3 = 5 + 17t + 36t^2 + 62t^3 + \cdots$$
$$h(M_4, t) = (2 - t)(1 + 2t + 2t^2)/(1 - t)^6 = 2 + 15t + 62t^2 + 185t^3 + \cdots$$

while for $\mathcal{A}_2$,

$$h(M_3, t) = (5 + t)/(1 - t)^3 = 5 + 16t + 33t^2 + 56t^3 + \cdots$$
$$h(M_4, t) = (1 + 6t - t^2 - t^3)/(1 - t)^6 = 1 + 12t + 56t^2 + 175t^3 + \cdots$$

Using formula (34), this yields expressions for the Hilbert series of $U(\mathfrak{g}_1)$ and $U(\mathfrak{g}_2)$. Comparing these Hilbert series shows $U(\mathfrak{g}_1) \ncong U(\mathfrak{g}_2)$, and hence the two arrangements must have non-isomorphic homotopy Lie algebras.

**Example 6.3.** In 1946, Nandi [19] showed that there are exactly three inequivalent block designs with parameters $(10, 15, 6, 4, 2)$. We list the blocks of each below. Each block design gives rise to a rank-4 matroid on ten points by taking the dependent sets to be those subsets that either contain one of the blocks or contain at least five elements.

| | |
|---|---|
| $D_1$ | {abcd, abef, aceg, adhi, bchi, bdgj, cdfj, afhj, agij, behj, bfgi, ceij, cfgh, defi, degh} |
| $D_2$ | {abcd, abef, aceg, adhi, bcij, bdgh, cdfj, afhj, agij, aehj, bfgi, cehi, cfgh, defi, degj} |
| $D_3$ | {abcd, abef, acgh, adij, bcij, bdgh, cdef, aegi, afhj, behj, bfgi, cehi, cfgj, degj, dfhi} |

By construction, there are no nontrivial, dependent sets of size three, so each arrangement is 2-generic.

If we call the corresponding Orlik-Solomon algebras $A_1$, $A_2$, and $A_3$, it is straightforward to calculate that $h(A_i, t) = 1 + 10t + 45t^2 + 105t^3 + 69t^4$ for $i = 1, 2, 3$. In each case, the singular range is $(3, 4)$. The ideals $J_1$, $J_2$, $J_3$ have differing resolutions, however, from which it follows that $\mathfrak{g}_{A_1}$, $\mathfrak{g}_{A_2}$, and $\mathfrak{g}_{A_3}$ are pairwise non-isomorphic.

## 7. Topological interpretations

7.1. **Generic slices.** A particularly simple situation, analyzed in detail by Dimca and Papadima in [6], is when $\mathcal{A}$ is a generic slice of rank $\ell > 2$ of a supersolvable arrangement $\mathcal{B}$. Let $\mathcal{A}'$ and $\mathcal{B}'$ be the respective decones, with complements $X = X(\mathcal{A}')$ and $Y = X(\mathcal{B}')$. The two spaces share the same fundamental group, $\pi$, and the same integral holonomy Lie algebra, $\mathfrak{h}$.

In [6, Theorems 18(ii) and 23], Dimca and Papadima establish the following facts. The universal enveloping algebra $U(\mathfrak{h})$ is isomorphic (as a Hopf algebra) to the associated graded algebra $\mathrm{gr}_{I\pi}(\mathbb{Z}\pi)$, where $\mathbb{Z}\pi$ is the group ring of $\pi$, with filtration determined by the powers of the augmentation ideal $I\pi$. The first non-vanishing higher homotopy group of $X$ is $\pi_{\ell-1}(X)$; when viewed as a module over $\mathbb{Z}\pi$, it has resolution of the form

$$0 \longrightarrow H_m(Y)\otimes\mathbb{Z}\pi \longrightarrow \cdots \longrightarrow H_\ell(Y)\otimes\mathbb{Z}\pi \longrightarrow \pi_{\ell-1}(X) \longrightarrow 0\ . \tag{45}$$

Finally, the associated graded module of $\pi_{\ell-1}(X)$, with respect to the filtration by powers of $I\pi$, has Hilbert series

$$h(\operatorname{gr}^\bullet_{I\pi}\pi_{\ell-1}(X),t) = (-1/t)^\ell\left(1-\frac{\sum_{j=0}^{\ell-1}(-1)^j\beta_j t^j}{\sum_{j=0}^{m}(-1)^j\beta_j t^j}\right), \tag{46}$$

where $\beta_j$ are the Betti numbers of $Y$.

Consider the integral cohomology rings $A = H^\bullet(X,\mathbb{Z})$ and $B = H^\bullet(Y,\mathbb{Z})$. We have $(B^i)^* = H_i(Y,\mathbb{Z})$, since the homology of an arrangement complement is torsion-free. Thus, tensoring with $\Bbbk$, and passing to the associated graded in resolution (45) recovers resolution (32). As a consequence, we obtain the following.

**Proposition 7.1.** *Let $\mathcal{A}$ be a generic slice of rank $\ell > 2$ of a supersolvable arrangement, and let $X = X(\mathcal{A}')$ be the complement of its decone. The homotopy module of the algebra $A = H^\bullet(X,\Bbbk)$ is isomorphic to the graded module associated to the the first nonvanishing higher homotopy group of $X$:*

$$M_A \cong \operatorname{gr}^\bullet_I \pi_{\ell-1}(X)\otimes\Bbbk. \tag{47}$$

7.2. **Rescaling.** Fix an integer $q \geq 1$. The $q$-rescaling of a graded algebra $A$ is the graded algebra $A^{[q]}$, with $A^{[q]}_{i(2q+1)} = A_i$ and $A^{[q]}_j = 0$ if $(2q+1)\nmid j$, and with multiplication rescaled accordingly. When taking the Yoneda algebra of $A^{[q]}$, the internal degree of the Yoneda algebra of $A$ gets rescaled, while the resolution degree stays unchanged:

$$\operatorname{Ext}_{A^{[q]}}(\Bbbk,\Bbbk) = \operatorname{Ext}_A(\Bbbk,\Bbbk)^{[q]}. \tag{48}$$

Similarly, the $q$-rescaling of a graded Lie algebra $L$ is the graded Lie algebra $L^{[q]}$, with $L^{[q]}_{2iq} = L_i$ and $L^{[q]}_j = 0$ if $2q \nmid j$, and with Lie bracket rescaled accordingly. Rescaling works well with the holonomy and homotopy Lie algebras:

$$\mathfrak{h}_{A^{[q]}} = \mathfrak{h}_A^{[q]}, \quad \mathfrak{g}_{A^{[q]}} = \mathfrak{g}_A^{[q]}. \tag{49}$$

The Hilbert series of the enveloping algebras of $\mathfrak{g}_A^{[q]}$ and $\mathfrak{g}_A$ are related as follows:

$$h(U(\mathfrak{g}_A^{[q]}),t,u) = h(U(\mathfrak{g}_A),tu^{2q},u^{2q+1}). \tag{50}$$

Now let $X$ be a connected, finite-type CW-complex. A simply-connected, finite-type CW-complex $Y$ is called a $q$-rescaling of $X$ (over a field $\Bbbk$) if the cohomology algebra $H^\bullet(Y,\Bbbk)$ is the $q$-rescaling of $H^\bullet(X,\Bbbk)$, i.e.,

$$H^\bullet(Y,\Bbbk) = H^\bullet(X,\Bbbk)^{[q]}. \tag{51}$$

Rational rescalings always exist: take a Sullivan minimal model for the 1-connected, finite-type differential graded algebra $(H^\bullet(X,\mathbb{Q})^{[q]}, d=0)$, and use [29] to realize it by a finite-type, 1-connected CW-complex, $Y$. The space constructed this way is the desired rescaling. Moreover, $Y$ is formal, i.e, its rational homotopy type is a formal consequence of its rational cohomology algebra. Hence, $Y$ is uniquely determined, up to rational homotopy equivalence, among spaces with the same cohomology ring (though there may be other, non-formal rescalings of $X$, see [22]).

**Proposition 7.2.** *Let $X$ be a finite-type CW-complex, with cohomology algebra $A = H^\bullet(X;\mathbb{Q})$. Let $Y$ be a finite-type, simply-connected CW-complex with $H^\bullet(Y;\mathbb{Q}) \cong A^{[q]}$. If $Y$ is formal, then*

$$\pi_\bullet(\Omega Y) \otimes \mathbb{Q} \cong \mathfrak{g}_A^{[q]}. \tag{52}$$

*Proof.* Since $Y$ is formal, the Eilenberg-Moore spectral sequence of the path fibration $\Omega Y \to PY \to Y$ collapses, yielding an isomorphism of Hopf algebras between the Yoneda algebra of $H^\bullet(Y;\mathbb{Q})$ and the Pontryagin algebra $H_\bullet(\Omega Y;\mathbb{Q})$. From the rescaling assumption, we obtain

$$\operatorname{Ext}_{A^{[q]}}(\mathbb{Q},\mathbb{Q}) \cong H_\bullet(\Omega Y;\mathbb{Q}), \tag{53}$$

By Milnor-Moore [18], we find that $\mathfrak{g}_{A^{[q]}} \cong \pi_\bullet(\Omega Y) \otimes \mathbb{Q}$, as Lie algebras. Using (49) finishes the proof. ☐

As a consequence, we obtain a quick proof of a special case of Theorem A from [22].

**Corollary 7.3** ([22])**.** *Suppose $X$ and $Y$ are spaces as above. If both $X$ and $Y$ are formal and $A$ is Koszul, then*

$$\pi_\bullet(\Omega Y) \otimes \mathbb{Q} \cong \left(\operatorname{gr}_\bullet(\pi_1 X) \otimes \mathbb{Q}\right)^{[q]}. \tag{54}$$

*Proof.* Since $A$ is Koszul, $\mathfrak{g}_A = \mathfrak{h}_A$. Since $X$ is formal, $\operatorname{gr}_\bullet(\pi_1 X) \otimes \mathbb{Q} \cong \mathfrak{h}_A$, cf. [29]. The conclusion follows from (52). ☐

**Remark 7.4.** When $X$ is formal (but not necessarily simply connected), a theorem of Papadima and Yuzvinsky [23] states that the cohomology algebra $A = H^\bullet(X;\mathbb{Q})$ is Koszul if and only if the Bousfield-Kan rationalization $X_{\mathbb{Q}}$ is aspherical. Now, by a classical result of Quillen [24], $U(\mathfrak{h}_A) \cong \operatorname{gr}_{I\pi} \mathbb{Q}\pi_1(X_{\mathbb{Q}})$. More generally, it seems likely that

$$U(\mathfrak{g}_A) \cong U(\pi_\bullet(\Omega \widetilde{X}_{\mathbb{Q}})) \mathbin{\widehat{\otimes}} \operatorname{gr}_{I\pi} \mathbb{Q}\pi_1(X_{\mathbb{Q}}), \tag{55}$$

in view of a result of Félix and Thomas [10]. (Here again, $\mathbb{Q}\pi_1(X_{\mathbb{Q}})$ acts on the left-hand factor by the action induced from $\pi_1(X_{\mathbb{Q}})$ on the universal cover $\widetilde{X_{\mathbb{Q}}}$.)

However, if $X$ is a hyperplane arrangement complement, then $X$ is not in general a nilpotent space. This means that we can expect to find such spaces $X$ for which $\pi_i(X_{\mathbb{Q}}) \ncong \pi_i(X) \otimes \mathbb{Q}$. The first such example was found by Falk [7], who noted that the complement $X$ of the $D_4$ reflection arrangement is aspherical, while its Bousfield-Kan rationalization $X_{\mathbb{Q}}$ is not. In general, then, we know of no way to relate $\mathfrak{g}_A$ with the topological homotopy Lie algebra, $\pi_\bullet(\Omega X) \otimes \mathbb{Q}$.

7.3. **Redundant subspace arrangements.** Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be an arrangement of hyperplanes in $\mathbb{C}^\ell$. If $q$ is a positive integer, then $\mathcal{A}^{(q)} = \{H_1^{\times q}, \dots, H_n^{\times q}\}$ is an arrangement of codimension $q$ subspaces in $\mathbb{C}^{q\ell}$. For example, if $\mathcal{A}$ is the braid arrangement in $\mathbb{C}^\ell$, with complement equal to the configuration space of $\ell$ distinct points in $\mathbb{C}$, then the complement of $\mathcal{A}^{(q)}$ is the configuration space of $\ell$ distinct points in $\mathbb{C}^q$.

**Proposition 7.5.** *Let $\mathcal{A}$ be a hyperplane arrangement, with Orlik-Solomon algebra $A = H^\bullet(X;\mathbb{Q})$. Fix $q \geq 1$, and let $Y = X(\mathcal{A}^{(q+1)})$ be the complement of the corresponding subspace arrangement. Then:*

$$\pi_\bullet(\Omega Y) \otimes \mathbb{Q} \cong \mathfrak{g}_A^{[q]}. \tag{56}$$

*Proof.* Clearly, $Y$ is simply-connected. As shown in [3], $H^\bullet(Y;\mathbb{Q})$ is the $q$-rescaling of $H^\bullet(X;\mathbb{Q})$. Since $\mathcal{A}^{(q+1)}$ has geometric intersection lattice, its complement $Y$ is formal, see [30, Prop. 7.2]. The conclusion then follows from Proposition 7.2. □

**Corollary 7.6.** *Let $\mathcal{A}$ be a hypersolvable arrangement, satisfying either of the hypothesis of Theorem 1.8. Then*

$$\pi_\bullet(\Omega Y) \otimes \mathbb{Q} \cong (\mathrm{Lie}(M_A[-2]) \rtimes \mathfrak{h}_A)^{[q]}.$$

**Example 7.7.** Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be the hyperplane arrangements from Example 6.2. Denote by $\mathfrak{g}_i = \mathfrak{g}_{A_i}$ the respective homotopy Lie algebras, $i = 1, 2$. Consider the redundant subspace arrangements $\mathcal{A}_1^{(2)}$ and $\mathcal{A}_2^{(2)}$. Both are arrangements of 7 codimension-2 complex subspaces of $\mathbb{C}^8$. Denoting their complements by $Y_1$ and $Y_2$, respectively, we have $\pi_1(Y_1) = \pi_1(Y_2) = 0$ and $H_*(Y_1) \cong H_*(Y_2)$ as graded abelian groups.

Let $\pi_\bullet(\Omega Y_i) \otimes \mathbb{Q}$ be the respective (topological) homotopy Lie algebras. By Proposition 7.2, we have $\pi_\bullet(\Omega Y_i) \otimes \mathbb{Q} \cong \mathfrak{g}_i^{[1]}$. Making use of the previous calculations for the arrangements $\mathcal{A}_1$ and $\mathcal{A}_2$, together with formula (50), we find that $U(\mathfrak{g}_i^{[1]})_p$ has rank $1, 0, 7, 0, 28, 0, 84, 5, 210$ for $0 \leq p \leq 8$, for both $i = 1, 2$. It follows that, for $p \leq 9$, the group $\pi_p(Y_i) \otimes \mathbb{Q} = 0$, except for $\pi_3(Y_i) \otimes \mathbb{Q} \cong \mathbb{Q}^7$ and $\pi_8(Y_i) \otimes \mathbb{Q} \cong \mathbb{Q}^5$.

However, for $p = 9$, the ranks of $U(\mathfrak{g}_i^{[1]})_p$ are 52 and 51, respectively. Hence,

$$\pi_{10}(Y_1) \otimes \mathbb{Q} \cong \mathbb{Q}^{17} \quad \text{and} \quad \pi_{10}(Y_2) \otimes \mathbb{Q} \cong \mathbb{Q}^{16},$$

and so $Y_1 \not\simeq Y_2$.

**Acknowledgment.** A substantial portion of this work was carried out while the authors were attending the program "Hyperplane Arrangements and Applications" at the Mathematical Sciences Research Institute in Berkeley, California. We thank MSRI for its support and hospitality during this stay.

## References

1. Luchezar Avramov, *Infinite free resolutions*, Six lectures on commutative algebra (Bellaterra, 1996), 1–118, Progr. Math., vol. 166, Birkhäuser, Basel, 1998. MR **99m:**13022
2. Anders Björner and Günter Ziegler, *Broken circuit complexes: Factorizations and generalizations*, J. Combin Theory Series B **51** (1991), no. 1, 96–126. MR **92b:**52027

3. Daniel C. Cohen, Frederick R. Cohen and Miguel Xicoténcatl, *Lie algebras associated to fiber-type arrangements*, Intern. Math. Res. Not. **2003:29** (2003), 1591–1621. MR **2005a:**17019
4. Graham Denham and Sergey Yuzvinsky, *Annihilators of Orlik-Solomon relations*, Adv. in Appl. Math. **28** (2002), no. 2, 231–249. MR **2003b:**05046
5. David Eisenbud, Sorin Popescu and Sergey Yuzvinsky, *Hyperplane arrangement cohomology and monomials in the exterior algebra*, Trans. Amer. Math. Soc. **355** (2003), no. 11, 4365–4383. MR **2004g:**52036
6. Alexandru Dimca and Stefan Papadima, *Hypersurface complements, Milnor fibers and higher homotopy groups of arrangments*, Ann. of Math. (2) **158** (2003), no. 2, 473–507. MR **2005a:**32028
7. Michael Falk, *The minimal model of the complement of an arrangement of hyperplanes*, Trans. Amer. Math. Soc. 309 (1988), no. 2, 543–556. MR **89d:**32024
8. Michael Falk, *Line-closed matroids, quadratic algebras, and formal arrangements*, Adv. in Appl. Math. **28** (2002), no. 2, 250–271. MR **2003a:**05040
9. Michael Falk and Richard Randell, *The lower central series of a fiber-type arrangement*, Invent. Math. **82** (1985), no. 1, 77–88. MR **87c:**32015b
10. Yves Félix and Jean-Claude Thomas, *On the ubiquity of the rational homotopy Lie algebra of a topological space*, Bull. Soc. Math. Belg. Sér. A **38** (1986), 175–190. MR **88f:**55015
11. Ralph Fröberg, *Koszul algebras*, Advances in commutative ring theory (Fez, 1997), 337–350, Lecture Notes in Pure and Appl. Math., vol. 205, Dekker, New York, 1999. MR **2001i:**16046
12. Ralph Fröberg and Clas Löfwall, *Koszul homology and Lie algebras with application to generic forms and points*, Homology Homotopy Appl. **4** (2002), no. 2, part 2, 227–258. MR **2003g:**13018
13. Dan Grayson and Michael Stillman, *Macaulay 2: a software system for research in algebraic geometry*; available at `http://www.math.uiuc.edu/Macaulay2`.
14. Michel Jambu and Stefan Papadima, *A generalization of fiber-type arrangements and a new deformation method*, Topology **37** (1998), no. 6, 1135–1164. MR **99g:**52019
15. Michel Jambu and Ştefan Papadima, *Deformations of hypersolvable arrangements*, Topology Appl. **118** (2002), no. 1-2, 103–111, Arrangements in Boston: a Conference on Hyperplane Arrangements (1999). MR **2003a:**32047
16. Peter Jørgensen, *Non-commutative Castelnuovo-Mumford regularity*, Math. Proc. Cambridge Philos. Soc. **125** (1999), no. 2, 203–221. MR **2000h:**13010
17. Clas Löfwall, *On the subalgebra generated by the one-dimensional elements in the Yoneda Ext-algebra*, Algebra, algebraic topology and their interactions (Stockholm, 1983), Lecture Notes in Math., vol. 1183, Springer, Berlin, 1986, pp. 291–338. MR **88f:**16030
18. John W. Milnor and John C. Moore, *On the structure of Hopf algebras*, Ann. of Math. (2) **81** (1965), 211–264. MR 30 #4259
19. Hari Kinkar Nandi, *Enumeration of non-isomorphic solutations of balanced incomplete block designs*, Sankhyā **7** (1946), 305–312. MR 8,127c
20. Peter Orlik and Hiroaki Terao, *Arrangements of hyperplanes*, Grundlehren der Mathematischen Wissenschaften, no. 300, Springer-Verlag, Berlin, 1992. MR **94e:**52014
21. Stefan Papadima and Alexander I. Suciu, *Higher homotopy groups of complements of hyperplane arrangements*, Advances in Math. **165** (2002), no. 1, 71–100. MR **2003b:**55019
22. Stefan Papadima and Alexander I. Suciu, *Homotopy Lie algebras, lower central series, and the Koszul property*, Geometry & Topology **8** (2004), 1079–1125. MR 2087079
23. Stefan Papadima and Sergey Yuzvinsky, *On rational $K[\pi, 1]$ spaces and Koszul algebras*, J. Pure Appl. Alg. **144** (1999), no. 2, 157–167. MR **2000k:**55017
24. Daniel Quillen, *On the associated graded ring of a group ring*, J. Algebra **10** (1968), 411–418. MR 38 #245
25. Vadim V. Schechtman and Alexander N. Varchenko, *Arrangements of hyperplanes and Lie algebra homology*, Invent. Math. **106** (1991), no. 1, 139–194. MR **93b:**17067
26. Henry K. Schenck and Alexander I. Suciu, *Resonance, linear syzygies, Chen groups, and the Bernstein-Gelfand-Gelfand correspondence*, to appear in Trans. Amer. Math. Soc.

27. David W. Sharpe and Peter Vámos, *Injective modules*, Cambridge Tracts in Mathematics and Mathematical Physics, no. 62, Cambridge University Press, London-New York, 1972. MR 50 #13153
28. Brad Shelton and Sergey Yuzvinsky, *Koszul algebras from graphs and hyperplane arrangements*, J. London Math. Soc. (2) **56** (1997), no. 3, 477–490. MR **99c:**16044
29. Dennis Sullivan, *Infinitesimal computations in topology*, Inst. Hautes Études Sci. Publ. Math. **47** (1977), 269–331. MR 58 #31119
30. Sergey Yuzvinsky, *Small rational model of subspace complement*, Trans. Amer. Math. Soc. **354** (2002), no. 5, 1921–1945 MR **2003a:**52030

Department of Mathematics, University of Western Ontario, London, ON N6A 5B7
*E-mail address*: `gdenham@uwo.ca`
*URL*: `http://www.math.uwo.ca/~gdenham`

Department of Mathematics, Northeastern University, Boston, MA 02115
*E-mail address*: `a.suciu@neu.edu`
*URL*: `http://www.math.neu.edu/~suciu`